Differential of the Exponential Map

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1 Introduction

This document computes

$$\left[\frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0}\right]\log\left(\exp\left(x+\epsilon\right)\cdot\exp\left(x\right)^{-1}\right) \tag{1}$$

where exp and log are the exponential mapping and its inverse in a Lie group, and x and ϵ are elements of the associated Lie algebra.

2 Definitions

Let $\mathcal{G}$ be a Lie group, with associated Lie algebra $\mathfrak{g}$. Then the exponential map takes algebra elements to group elements:

$$\exp:\mathfrak{g} \rightarrow \mathcal{G} \tag{2}$$

$$\exp\left(x\right) = \mathbf{I}+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\ldots \tag{3}$$

The adjoint representation Adj of the group linearly transforms the exponential mapping of an algebra element through left multiplication by a group element:

$$x \in \mathfrak{g} \tag{4}$$

$$Y \in \mathcal{G} \tag{5}$$

$$Y\cdot\exp\left(x\right) = \exp\left(\mathrm{Adj}_Y\cdot x\right)\cdot Y \tag{6}$$

The adjoint operator in the algebra is the linear operator representing the Lie bracket:

$$x,y \in \mathfrak{g} \tag{7}$$

$$\mathrm{ad}_x\cdot y = x\cdot y-y\cdot x \tag{8}$$

The adjoint operator commutes with the exponential map:

$$\mathrm{Adj}_{\exp(y)} = \exp\left(\mathrm{ad}_y\right) \tag{9}$$

We define differentiation of a function f from algebra to group as follows:

$$f : \mathfrak{g} \rightarrow \mathcal{G} \tag{10}$$

$$\frac{\partial f(x)}{\partial x} : \mathfrak{g} \rightarrow \mathfrak{g} \tag{11}$$

$$\frac{\partial f(x)}{\partial x} \equiv \left[\frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0}\right] \log\left(f(x+\epsilon) \cdot f(x)^{-1}\right) \tag{12}$$

In this document, we're interested in $D_{\exp}$, the differential of exp:

$$D_{\exp} : \mathfrak{g} \rightarrow \mathfrak{g} \tag{13}$$

$$D_{\exp}(x) = \frac{\partial \exp(x)}{\partial x} \tag{14}$$

3 Derivation of a formula for $D_{\exp}(x)$

This isn't a rigorous derivation (the epsilon-delta proofs required for the two approximation steps are omitted), but I find it intuitively pleasing. A more rigorous approach would use theorems about integrated flows on continuous vector fields.

Define F to be exp of x modified by an algebra element ϵ:

$$\epsilon \in \mathfrak{g} \tag{15}$$

$$F(x,\epsilon) = \exp(x+\epsilon) \tag{16}$$

We can also take the product of multiple smaller group elements on the same geodesic:

$$F(x,\epsilon) = \prod_{i=1}^{N} \exp\left(\frac{1}{N} \cdot (x+\epsilon)\right) \tag{17}$$

Letting the number of steps N go arbitrarily large, we can send $\frac{1}{N^2} \rightarrow 0$. Then we have, to arbitrary accuracy:

$$F(x,\epsilon) \approx \prod_{i=1}^{N} \exp\left(\frac{x}{N}\right) \cdot \exp\left(\frac{\epsilon}{N}\right) \tag{18}$$

Each factor of $\exp\left(\frac{\epsilon}{N}\right)$ can be shifted to the left side of the product by multiplying by the adjoint an appropriate number of times:

$$
\begin{aligned}
A_N &\equiv \mathrm{Adj}_{\exp\left(\frac{x}{N}\right)} && (19)\\
F(x,\epsilon) &\approx \left[\exp\left(\frac{1}{N}\cdot A_N\cdot\epsilon\right)\cdot\exp\left(\frac{1}{N}\cdot A_N^2\cdot\epsilon\right)\cdot\ldots\cdot\exp\left(\frac{1}{N}\cdot A_N^N\cdot\epsilon\right)\right]\cdot\left[\prod_{i=1}^{N}\exp\left(\frac{x}{N}\right)\right] && (20)\\
&= \left[\prod_{i=1}^{N}\exp\left(\frac{1}{N}\cdot A_N^i\cdot\epsilon\right)\right]\cdot\left[\prod_{i=1}^{N}\exp\left(\frac{x}{N}\right)\right] && (21)\\
&= \left[\prod_{i=1}^{N}\exp\left(\frac{1}{N}\cdot A_N^i\cdot\epsilon\right)\right]\cdot\exp(x) && (22)
\end{aligned}
$$

By choosing ϵ sufficiently small, the product of exponentials is arbitrarily well approximated by the exponential of a sum:

$$
F(x,\epsilon)=\exp\left(\frac{1}{N}\cdot\sum_{i=1}^{N}A_N^i\cdot\epsilon+O\left(\|\epsilon\|^2\right)\right)\cdot\exp(x) \tag{23}
$$

We can use the properties of the adjoint to rewrite A_N:

$$
\begin{aligned}
A_N &\equiv \mathrm{Adj}_{\exp\left(\frac{x}{N}\right)} && (24)\\
&= \exp\left(\mathrm{ad}_{\frac{x}{N}}\right) && (25)\\
&= \exp\left(\frac{1}{N}\cdot\mathrm{ad}_x\right) && (26)
\end{aligned}
$$

for a Lie group

Taking the i^{th} power:

$$
A_N^i=\exp\left(\frac{i}{N}\cdot\mathrm{ad}_x\right) \tag{27}
$$

Thus as $N\to\infty$, the sum becomes an integral:

$$
\begin{aligned}
\frac{1}{N}\cdot\sum_{i=1}^{N}A_N^i &= \frac{1}{N}\cdot\sum_{i=1}^{N}\exp\left(\frac{i}{N}\cdot\mathrm{ad}_x\right) && (28)\\
&\to \int_0^1\exp\left(t\cdot\mathrm{ad}_x\right)\cdot\mathrm{d}t && (29)
\end{aligned}
$$

The integration can be performed on the power series of the matrix exponential.

$$
\begin{aligned}
\frac{1}{N}\cdot\sum_{i=1}^{N}A_N^i &= \int_0^1\left(\sum_{i=0}^{\infty}\frac{t^i\cdot\mathrm{ad}_x^i}{i!}\right)\cdot\mathrm{d}t && (30)\\
&= \left(\sum_{i=0}^{\infty}\frac{t^{i+1}\mathrm{ad}_x^i}{(i+1)!}\right)\Big|_0^1 && (31)\\
&= \sum_{i=0}^{\infty}\frac{\mathrm{ad}_x^i}{(i+1)!} && (32)
\end{aligned}
$$

Substituting into Eq.23:

$$F(x,\epsilon) = \exp\left(\left(\sum_{i=0}^{\infty} \frac{\mathrm{ad}_x^i}{(i+1)!}\right)\cdot\epsilon + O\left(\|\epsilon\|^2\right)\right)\cdot\exp(x)$$

Using the definition from Eq.14,

$$D_{\exp}(x) = \left[\frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0}\right]\log\left(F(x,\epsilon)\cdot\exp(x)^{-1}\right) \quad (33)$$

$$= \left[\frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0}\right]\left(\sum_{i=0}^{\infty}\frac{\mathrm{ad}_x^i}{(i+1)!}\right)\cdot\epsilon + O\left(\|\epsilon\|^2\right) \quad (34)$$

$$= \sum_{i=0}^{\infty}\frac{\mathrm{ad}_x^i}{(i+1)!} \quad (35)$$

4 Differential of log

When $x = \log(\exp(x))$, we can invert the function being differentiated in Eq.14:

$$\delta \equiv f(\epsilon) = \log\left(\exp(x+\epsilon)\cdot\exp(x)^{-1}\right) \quad (36)$$

$$\epsilon = \log(\exp(\delta)\cdot\exp(x)) - x \quad (37)$$

The second term vanishes when differentiating by δ:

$$D_{\log}(x) \equiv \left[\frac{\partial}{\partial\delta}\Big|_{\delta=0}\right]\log(\exp(\delta)\cdot\exp(x)) \quad (38)$$

In this bijective region of the function, the differential of the inverse is the inverse of the differential:

$$\frac{\partial\epsilon}{\partial\delta} = \left[\frac{\partial\delta}{\partial\epsilon}\right]^{-1} \quad (39)$$

$$D_{\log}(x) = D_{\exp}^{-1}(x) \quad (40)$$

5 Special Cases

The infinite series of Eq. 35 can be expressed in closed form in some Lie groups.

5.1 SO(3)

5.1.1 Differential of exp

The elements of the algebra $\mathfrak{so}(3)$ are 3×3 skew-symmetric matrices, and the adjoint representation is identical:

$$\begin{aligned}
\omega &\in \Re^3 && (41)\\
\omega_\times &= \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \in \mathfrak{so}(3) && (42)\\
\mathrm{ad}_\omega &= \omega_\times && (43)\\
\mathrm{ad}_\omega^3 &= -\left\|\omega\right\|^2 \cdot \mathrm{ad}_\omega && (44)
\end{aligned}$$

Because the higher powers of ad collapse back to lower powers, we can collect terms in the series:

$$\begin{aligned}
D_{\exp}\left(\omega\right) &= \mathbf{I} + \left(\sum_{i=0}^{\infty} \frac{(-1)^i \cdot \left\|\omega\right\|^{2i}}{(2i+2)!}\right) \cdot \mathrm{ad}_\omega + \left(\sum_{i=0}^{\infty} \frac{(-1)^i \cdot \left\|\omega\right\|^{2i}}{(2i+3)!}\right) \cdot \mathrm{ad}_\omega^2 && (45)\\
&= \mathbf{I} + \left(\frac{1-\cos\left\|\omega\right\|}{\left\|\omega\right\|^2}\right) \cdot \omega_\times + \left(\frac{1-\frac{\sin\left\|\omega\right\|}{\left\|\omega\right\|}}{\left\|\omega\right\|^2}\right) \cdot \omega_\times^2 && (46)
\end{aligned}$$

Note that

$$\omega_\times^2 = \omega\omega^T - \left\|\omega\right\|^2 \mathbf{I} \qquad (47)$$

So $D_{\exp}\left(\omega\right)$ can be rewritten:

$$\begin{aligned}
D_{\exp}\left(\omega\right) &= \mathbf{I} + \left(\frac{1-\cos\left\|\omega\right\|}{\left\|\omega\right\|^2}\right) \cdot \omega_\times + \left(\frac{1-\frac{\sin\left\|\omega\right\|}{\left\|\omega\right\|}}{\left\|\omega\right\|^2}\right) \cdot \left(\omega\omega^T - \left\|\omega\right\|^2 \mathbf{I}\right) && (48)\\
&= \frac{\sin\left\|\omega\right\|}{\left\|\omega\right\|} \cdot \mathbf{I} + \left(\frac{1-\cos\left\|\omega\right\|}{\left\|\omega\right\|^2}\right) \cdot \omega_\times + \left(\frac{1-\frac{\sin\left\|\omega\right\|}{\left\|\omega\right\|}}{\left\|\omega\right\|^2}\right) \cdot \omega\omega^T && (49)
\end{aligned}$$

We label the coefficients for convenience:

$$\begin{aligned}
a_\theta &= \frac{\sin\theta}{\theta} && (50)\\
b_\theta &= \frac{1-\cos\theta}{\theta^2} && (51)\\
c_\theta &= \frac{1-a_\theta}{\theta^2} && (52)\\
D_{\exp}\left(\omega\right) &= a_{\left\|\omega\right\|} \cdot \mathbf{I} + b_{\left\|\omega\right\|} \cdot \omega_\times + c_{\left\|\omega\right\|} \cdot \omega\omega^T && (53)
\end{aligned}$$

5.1.2 Differential of log

Recall that in the bijective region of exp and log,

$$D_{\log}(\omega) = D_{\exp}^{-1}(\omega) \tag{54}$$

For $\|\omega\| < 2\pi$, a closed-form inverse exists for $D_{\exp}(\omega)$:

$$\begin{aligned} D_{\exp}^{-1}(\omega) &= \mathbf{I} - \frac{1}{2}\omega_\times + e_{\|\omega\|}\omega_\times^2 && (55)\\ e_\theta &= \frac{b_\theta - 2c_\theta}{2a_\theta} && (56)\\ &= \frac{b_\theta - \frac{1}{2}a_\theta}{1-\cos\theta} && (57) \end{aligned}$$

Depending on the value of θ, the more convenient of Eq. 56 or Eq.57 should be used to compute e_θ.

5.2 SE(3)

5.2.1 Differential of exp

Again, the higher powers of ad can be expressed in terms of lower powers:

$$\begin{aligned} u, \omega &\in \Re^3 && (58)\\ \theta &\equiv \|\omega\| && (59)\\ x &= \begin{pmatrix} \omega_\times & u \\ 0 & 0 \end{pmatrix} \in \mathfrak{se}(3) && (60)\\ \mathrm{ad}_x &= \begin{pmatrix} \omega_\times & u_\times \\ 0 & \omega_\times \end{pmatrix} && (61)\\ \mathrm{ad}_x^2 &= \begin{pmatrix} \omega_\times^2 & (\omega_\times u_\times + u_\times \omega_\times) \\ 0 & \omega_\times^2 \end{pmatrix} && (62)\\ \mathrm{ad}_x^3 &= -\theta^2 \cdot \mathrm{ad}_x - 2\left(\omega^T u\right) \begin{pmatrix} 0 & \omega_\times \\ 0 & 0 \end{pmatrix} && (63) \end{aligned}$$

Collecting the terms, we have:

$$\begin{aligned} Q(\omega) &\equiv \left(\frac{a_\theta - 2b_\theta}{\theta^2}\right) \cdot \omega_x + \left(\frac{b_\theta - 3c_\theta}{\theta^2}\right) \cdot \omega_\times^2 && (64)\\ D_{\exp}(x) &= \mathbf{I} + a_\theta \cdot \mathrm{ad}_x + c_\theta \cdot \mathrm{ad}_x^2 + \left(\omega^T u\right) \cdot \begin{pmatrix} 0 & Q(\omega) \\ 0 & 0 \end{pmatrix} && (65)\\ &= \begin{pmatrix} D_{\exp}(\omega) & \left(b_\theta \cdot u_\times + c_\theta \cdot (\omega_\times u_\times + u_\times \omega_\times) + \left(\omega^T u\right) \cdot Q(\omega)\right) \\ 0 & D_{\exp}(\omega) \end{pmatrix} && (66) \end{aligned}$$

Using the identity

$$\omega_\times u_\times + u_\times \omega_\times = \omega u^T + u\omega^T - 2\left(\omega^T u\right)\mathbf{I} \tag{67}$$

...we can rewrite $D_{\exp}(x)$:

$$\begin{aligned}
W(\omega) &\equiv -2c_\theta \cdot \mathbf{I} + Q(\omega) &(68)\\
&= -2c_\theta \cdot \mathbf{I} + \left(\frac{a_\theta - 2b_\theta}{\theta^2}\right)\cdot\omega_\times + \left(\frac{b_\theta - 3c_\theta}{\theta^2}\right)\cdot\left(\omega\omega^T - \theta^2\mathbf{I}\right) &(69)\\
&= (c_\theta - b_\theta)\cdot\mathbf{I} + \left(\frac{a_\theta - 2b_\theta}{\theta^2}\right)\cdot\omega_\times + \left(\frac{b_\theta - 3c_\theta}{\theta^2}\right)\cdot\omega\omega^T &(70)\\
D_{\exp}(x) &= \begin{pmatrix} D_{\exp}(\omega) & \left(b_\theta \cdot u_\times + c_\theta \cdot \left(\omega u^T + u\omega^T\right) + \left(\omega^T u\right)\cdot W(\omega)\right) \\ 0 & D_{\exp}(\omega)\end{pmatrix} &(71)
\end{aligned}$$

5.2.2 Differential of log

A square block matrix M with the form -

$$M = \begin{pmatrix} A & B \\ 0 & A \end{pmatrix} \tag{72}$$

...has an inverse:

$$M^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}\cdot B\cdot A^{-1} \\ 0 & A^{-1}\end{pmatrix} \tag{73}$$

Thus, when $\|\omega\| < 2\pi$, a closed form exists for $D^{-1}_{\exp}(x)$ using $D^{-1}_{\exp}(\omega)$ as given by Eq.55:

$$\begin{aligned}
B &\equiv b_\theta \cdot u_\times + c_\theta \cdot \left(\omega u^T + u\omega^T\right) + \left(\omega^T u\right)\cdot W(\omega) &(74)\\
D^{-1}_{\exp}(x) &= \begin{pmatrix} D^{-1}_{\exp}(\omega) & -D^{-1}_{\exp}(\omega)\cdot B\cdot D^{-1}_{\exp}(\omega) \\ 0 & D^{-1}_{\exp}(\omega)\end{pmatrix} &(75)
\end{aligned}$$