Lie Groups for 2D and 3D Transformations

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1 Introduction

This document derives useful formulae for working with the Lie groups that represent transformations in 2D and 3D space. A Lie group is a topological group that is also a smooth manifold, with some other nice properties. Associated with every Lie group is a Lie algebra, which is a vector space discussed below. Importantly, a Lie group and its Lie algebra are intimately related, allowing calculations in one to be mapped usefully into the other.

This document does not give a rigorous introduction to Lie groups, nor does it discuss all of the mathematical details of Lie groups in general. It does attempt to provide enough information that the Lie groups representing spatial transformations can be employed usefully in robotics and computer vision.

Here are the Lie groups that this document addresses:

Group	Description	Dim.	Matrix Representation
SO(3)	3D Rotations	3	3D rotation matrix
SE(3)	3D Rigid transformations	6	Linear transformation on homogeneous 4-vectors
SO(2)	2D Rotations	1	2D rotation matrix
SE(2)	2D Rigid transformations	3	Linear transformation on homogeneous 3-vectors
Sim(3)	3D Similarity transformations (rigid motion + scale)	7	Linear transformation on homogeneous 4-vectors

For each of these groups, the representation is described, and the exponential map and adjoint are derived.

1.1 Why use Lie groups for robotics or computer vision?

Many problems in robotics and computer vision involve manipulation and estimation in the 3D geometry. Without a coherent and robust framework for representing and working with 3D transformations,

*Added §2.4.2 to clarify the notation for differentiation of/by group element perturbations.

these tasks are onerous and treacherous. Transformations must be composed, inverted, differentiated and interpolated. Lie groups and their associated machinery address all of these operations, and do so in a principled way, so that once intuition is developed, it can be followed with confidence.

1.2 Lie algebras and other general properties

Every Lie group has an associated Lie algebra, which is the *tangent space* around the identity element of the group. That is, the Lie algebra is a vector space generated by differentiating the group transformations along chosen directions in the space, at the identity transformation. The tangent space has the same structure at all group elements, though tangent vectors undergo a coordinate transformation when moved from one tangent space to another. The basis elements of the Lie algebra (and thus of the tangent space) are called *generators* in this document. All tangent vectors represent linear combinations of the generators.

Importantly, the tangent space associated with a Lie group provides an "optimal" space in which to represent differential quantities related to the group. For instance, velocities, Jacobians, and covariances of transformations are well-represented in the tangent space around a transformation. This is the "optimal" space in which to represent differential quantities because

- The tangent space is a vector space with the same dimension as the number of degrees of freedom of the group transformations
- The exponential map converts any element of the tangent space *exactly* into a transformation in the group
- The adjoint *linearly* and *exactly* transforms tangent vectors from one tangent space to another

The adjoint property is what ensures that the tangent space has the same structure at all points on the manifold, because a tangent vector can always be transormed back to the tangent space around the identity.

Each Lie group described below also has a *group action* on 3D space. For instance, 3D rigid transformations have the action of rotating and translating points. The matrix representations given below make these actions explicit.

2 SO(3): Rotations in 3D space

2.1 Representation

Elements of the 3D rotation group, SO(3), are represented by 3D rotation matrices. Composition and inversion in the group correspond to matrix multiplication and inversion. Because rotation matrices are orthogonal, inversion is equivalent to transposition.

$$\mathbf{R} \in \mathrm{SO}(3) \tag{1}$$
$$\mathbf{R}^{-1} = \mathbf{R}^T \tag{2}$$

The Lie algebra, so(3), is the set of 3×3 skew-symmetric matrices. The generators of so(3) correspond to the derivatives of rotation around the each of the standard axes, evaluated at the identity:

$$G_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \; G_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \; G_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3}$$

An element of so(3) is then represented as a linear combination of the generators:

$$\boldsymbol{\omega} \in \mathbb{R}^3 \tag{4}$$

$$\boldsymbol{\omega}_1 G_1 + \boldsymbol{\omega}_2 G_2 + \boldsymbol{\omega}_3 G_3 \in \mathrm{so}(3) \tag{5}$$

We will simply write $\boldsymbol{\omega} \in \mathrm{so}(3)$ as a 3-vector of the coefficients, and use $\boldsymbol{\omega}_\times$ to represent the corresponding skew symmetric matrix.

2.2 Exponential Map

The exponential map that takes skew symmetric matrices to rotation matrices is simply the matrix exponential over a linear combination of the generators:

$$\exp(\boldsymbol{\omega}_\times) \equiv \exp\begin{pmatrix} 0 & -\boldsymbol{\omega}_3 & \boldsymbol{\omega}_2 \\ \boldsymbol{\omega}_3 & 0 & -\boldsymbol{\omega}_1 \\ -\boldsymbol{\omega}_2 & \boldsymbol{\omega}_1 & 0 \end{pmatrix} \tag{6}$$

$$= \mathbf{I} + \boldsymbol{\omega}_\times + \frac{1}{2!}\boldsymbol{\omega}_\times^2 + \frac{1}{3!}\boldsymbol{\omega}_\times^3 + \cdots \tag{7}$$

Writing the terms in pairs, we have:

$$\exp(\boldsymbol{\omega}_\times) = \mathbf{I} + \sum_{i=0}^{\infty}\left[\frac{\boldsymbol{\omega}_\times^{2i+1}}{(2i+1)!} + \frac{\boldsymbol{\omega}_\times^{2i+2}}{(2i+2)!}\right] \tag{8}$$

Now we can take advantage of a property of skew-symmetric matrices:

$$\boldsymbol{\omega}_\times^3 = -\left(\boldsymbol{\omega}^T\boldsymbol{\omega}\right) \cdot \boldsymbol{\omega}_\times \tag{9}$$

First extend this identity to the general case:

$$\theta^2 \equiv \boldsymbol{\omega}^T\boldsymbol{\omega} \tag{10}$$

$$\boldsymbol{\omega}_\times^{2i+1} = (-1)^i\theta^{2i}\boldsymbol{\omega}_\times \tag{11}$$

$$\boldsymbol{\omega}_\times^{2i+2} = (-1)^i\theta^{2i}\boldsymbol{\omega}_\times^2 \tag{12}$$

Now we can factor the exponential map series and recognize the Taylor expansions in the coefficients:

$$
\begin{aligned}
\exp\left(\boldsymbol{\omega}_\times\right) &= \mathbf{I} + \left(\sum_{i=0}^{\infty}\frac{(-1)^i\theta^{2i}}{(2i+1)!}\right)\boldsymbol{\omega}_\times + \left(\sum_{i=0}^{\infty}\frac{(-1)^i\theta^{2i}}{(2i+2)!}\right)\boldsymbol{\omega}_\times^2 &(13)\\
&= \mathbf{I} + \left(1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} + \cdots\right)\boldsymbol{\omega}_\times + \left(\frac{1}{2!} - \frac{\theta^2}{4!} + \frac{\theta^4}{6!} + \cdots\right)\boldsymbol{\omega}_\times^2 &(14)\\
&= \mathbf{I} + \left(\frac{\sin\theta}{\theta}\right)\boldsymbol{\omega}_\times + \left(\frac{1-\cos\theta}{\theta^2}\right)\boldsymbol{\omega}_\times^2 &(15)
\end{aligned}
$$

Equation 15 is the familiar Rodrigues formula. The exponential map yields a rotation by θ radians around the axis given by $\boldsymbol{\omega}$. Practical implementation of the Rodrigues formula should use the Taylor expansions of the coefficients of the second and third terms when θ is small.

The exponential map can be inverted to give the logarithm, going from SO(3) to so(3):

$$
\begin{aligned}
\mathbf{R} &\in \mathrm{SO}(3) &(16)\\
\theta &= \arccos\left(\frac{\mathrm{tr}(\mathbf{R})-1}{2}\right) &(17)\\
\ln\left(\mathbf{R}\right) &= \frac{\theta}{2\sin\theta}\cdot\left(\mathbf{R}-\mathbf{R}^T\right) &(18)
\end{aligned}
$$

The vector $\boldsymbol{\omega}$ is then taken from the off-diagonal elements of $\ln(\mathbf{R})$. Again, the Taylor expansion of the coefficient $\frac{\theta}{2\sin\theta}$ should be used when θ is small.

2.3 Adjoint

In Lie groups, it is often necessary to transform a tangent vector from the tangent space around one element to the tangent space of another. The *adjoint* performs this transformation. One very nice property of Lie groups in general is that this transformation is linear. For an element X of a Lie group, the adjoint is written Adj_X:

$$
\begin{aligned}
\boldsymbol{\omega} &\in \mathrm{so}(3),\ \mathbf{R} \in \mathrm{SO}(3) &(19)\\
\mathbf{R}\cdot\exp\left(\boldsymbol{\omega}\right) &= \exp\left(\mathrm{Adj}_{\mathbf{R}}\cdot\boldsymbol{\omega}\right)\cdot\mathbf{R} &(20)
\end{aligned}
$$

The adjoint can be computed from the generators of the Lie algebra:

$$
\begin{aligned}
\exp\left(\mathrm{Adj}_{\mathbf{R}}\cdot\boldsymbol{\omega}\right) &= \mathbf{R}\cdot\exp\left(\boldsymbol{\omega}\right)\cdot\mathbf{R}^{-1} &(21)\\
\mathrm{Adj}_{\mathbf{R}}\cdot\boldsymbol{\omega} &= \mathbf{R}\cdot\left(\sum_{i=1}^{3}\boldsymbol{\omega}_i G_i\right)\cdot\mathbf{R}^{-1} &(22)\\
&= \mathbf{R}\cdot\boldsymbol{\omega}_\times\cdot\mathbf{R}^{-1} &(23)\\
&= \left(\mathbf{R}\boldsymbol{\omega}\right)_\times &(24)\\
\Longrightarrow \mathrm{Adj}_{\mathbf{R}} &= \mathbf{R} &(25)
\end{aligned}
$$

In the case of SO(3), the adjoint transformation for an element is particularly simple: it is the same rotation matrix used to represent the element. Rotating a tangent vector by an element "moves" it from the tangent space on the right side of the element to the tangent space on the left.

2.4 Jacobians

2.4.1 Differentiating the action of SO(3) on $\mathcal{R}^3$

Consider $\mathbf{R} \in$SO(3) and $\mathbf{x} \in \mathcal{R}^3$. The rotation of vector $\mathbf{x}$ by matrix $\mathbf{R}$ is given by multiplication:

$$\mathbf{y} = f(\mathbf{R}, \mathbf{x}) = \mathbf{R} \cdot \mathbf{x} \tag{26}$$

Then differentiation by the vector is straightforward, as f is linear in $\mathbf{x}$:

$$\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \mathbf{R} \tag{27}$$

Differentiation by the rotation parameters is performed by implicitly left multiplying the rotation by the exponential of a tangent vector and differentiating the resulting expression around the zero perturbation. This is equivalent to left multiplying the product by the generators.

$$\begin{aligned}
\frac{\partial \mathbf{y}}{\partial \mathbf{R}} &= \frac{\partial}{\partial \boldsymbol{\omega}}|_{\boldsymbol{\omega}=\mathbf{0}} \left(\exp\left(\boldsymbol{\omega}\right) \cdot \mathbf{R}\right) \cdot \mathbf{x} && (28)\\
&= \frac{\partial}{\partial \boldsymbol{\omega}}|_{\boldsymbol{\omega}=\mathbf{0}} \exp\left(\boldsymbol{\omega}\right) \cdot \left(\mathbf{R} \cdot \mathbf{x}\right) && (29)\\
&= \frac{\partial}{\partial \boldsymbol{\omega}}|_{\boldsymbol{\omega}=\mathbf{0}} \exp\left(\boldsymbol{\omega}\right) \cdot \mathbf{y} && (30)\\
&= \left(\begin{array}{c|c|c} G_1\mathbf{y} & G_2\mathbf{y} & G_3\mathbf{y} \end{array}\right) && (31)\\
&= -\mathbf{y}_\times && (32)
\end{aligned}$$

2.4.2 Differentiating a group-valued function by an argument in the group

Consider a Lie group G and a function $f : G \to G$. Neither the domain nor the range is a vector space, but by introducing tangent space perturbations on the argument and result, we can use the differentation notation as a shorthand for the mapping from input to output perturbations:

$$\begin{aligned}
\exp\left(\boldsymbol{\epsilon}\right) \cdot f\left(g\right) &= f\left(\exp\left(\boldsymbol{\delta}\right) \cdot g\right) && (33)\\
\frac{\partial f}{\partial g} &\equiv \frac{\partial \boldsymbol{\epsilon}}{\partial \boldsymbol{\delta}}|_{\boldsymbol{\delta}=\mathbf{0}} && (34)
\end{aligned}$$

Solving Eq. 33 for $\boldsymbol{\epsilon}$ and differentiating yields an explicit formula for the differential of the output perturbation $\boldsymbol{\epsilon}$ by the input perturbation $\boldsymbol{\delta}$:

$$\boldsymbol{\epsilon} = \log\left(f\left(\exp\left(\boldsymbol{\delta}\right)\cdot g\right)\cdot f\left(g\right)^{-1}\right) \tag{35}$$

$$\frac{\partial f}{\partial g} \equiv \frac{\partial \log\left(f\left(\exp\left(\boldsymbol{\delta}\right)\cdot g\right)\cdot f\left(g\right)^{-1}\right)}{\partial \boldsymbol{\delta}}|_{\boldsymbol{\delta}=0} \tag{36}$$

Eq. 36 produces a linear mapping from left-tangent-space perturbations of the argument to left-tangent-space perturbations of the result. As expected, applying this differentiation shorthand to the identity function $f\left(g\right) = g$ yields the identity matrix.

For a nontrivial example application of this procedure, consider a product of elements in $G = \mathrm{SO}(3)$ by the second factor $\mathbf{R}_0$:

$$\mathbf{R}_2 = f\left(\mathbf{R}_0\right) \equiv \mathbf{R}_1 \cdot \mathbf{R}_0 \tag{37}$$

First, the input and output perturbations in the tangent space so(3) are made explicit.

$$\exp\left(\boldsymbol{\epsilon}\right)\cdot \mathbf{R}_2 = \mathbf{R}_1 \cdot \exp\left(\boldsymbol{\omega}\right)\cdot \mathbf{R}_0 \tag{38}$$

Differentation of $\boldsymbol{\epsilon}$ by the input perturbation $\boldsymbol{\omega}$ is performed around $\boldsymbol{\omega} = \mathbf{0}$. The adjoint is employed to shift the tangent vector to the left side of the expression. The remainder of the expression cancels and the result is simple.

$$\begin{aligned}
\frac{\partial \mathbf{R}_2}{\partial \mathbf{R}_0} &\equiv \frac{\partial \log\left(\left(\mathbf{R}_1\cdot\exp\left(\boldsymbol{\omega}\right)\cdot\mathbf{R}_0\right)\cdot\left(\mathbf{R}_1\cdot\mathbf{R}_0\right)^{-1}\right)}{\partial \boldsymbol{\omega}}|_{\boldsymbol{\omega}=0} && (39)\\
&= \frac{\partial}{\partial \boldsymbol{\omega}}|_{\boldsymbol{\omega}=0}\left[\log\left(\left(\exp\left(\mathrm{Adj}_{\mathbf{R}_1}\cdot\boldsymbol{\omega}\right)\cdot\mathbf{R}_1\cdot\mathbf{R}_0\right)\cdot\left(\mathbf{R}_1\cdot\mathbf{R}_0\right)^{-1}\right)\right] && (40)\\
&= \frac{\partial}{\partial \boldsymbol{\omega}}|_{\boldsymbol{\omega}=0}\left[\log\left(\exp\left(\mathrm{Adj}_{\mathbf{R}_1}\cdot\boldsymbol{\omega}\right)\right)\right] && (41)\\
&= \frac{\partial}{\partial \boldsymbol{\omega}}|_{\boldsymbol{\omega}=0}\left[\mathrm{Adj}_{\mathbf{R}_1}\cdot\boldsymbol{\omega}\right] && (42)\\
&= \mathrm{Adj}_{\mathbf{R}_1} && (43)\\
&= \mathbf{R}_1 && (44)
\end{aligned}$$

2.5 Gaussians in SO(3)

2.5.1 Sampling

We can encode Gaussian distributions over 3D rotations by representing the mean with an element of SO(3) and the covariance as a quadratic form over tangent vectors in so(3). More precisely, consider a Gaussian distribution given by mean $\mathbf{R} \in$SO(3) and covariance $\boldsymbol{\Sigma} \in \mathcal{R}^{3\times 3}$. We can draw a sample rotation $\mathbf{S}$ from the distribution by sampling the zero-mean distribution in the tangent space and left multiplying the mean:

$$\begin{aligned}\boldsymbol{\epsilon} &\in \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}) &(45)\\ \mathbf{S} &= \exp(\boldsymbol{\epsilon}) \cdot \mathbf{R} &(46)\end{aligned}$$

2.5.2 Composition of uncertain rotations

Given two Gaussian distributions on rotation, we can compose the two uncertain transformations using the adjoint. Let one mean-covariance pair be $(\mathbf{R}_0, \boldsymbol{\Sigma}_0)$ and the other be $(\mathbf{R}_1, \boldsymbol{\Sigma}_1)$. Then the distribution of rotations by first transforming by $\mathbf{R}_0$ and then by $\mathbf{R}_1$ is given by:

$$(\mathbf{R}_1, \boldsymbol{\Sigma}_1) \circ (\mathbf{R}_0, \boldsymbol{\Sigma}_0) = \left(\mathbf{R}_1 \cdot \mathbf{R}_0, \boldsymbol{\Sigma}_1 + \mathbf{R}_1 \cdot \boldsymbol{\Sigma}_0 \cdot \mathbf{R}_1^T\right) \tag{47}$$

2.5.3 Bayesian combination of rotation estimates

The information from the two Gaussians can be combined in a Bayesian manner to yield $(\mathbf{R}_c, \boldsymbol{\Sigma}_c)$ by first finding the deviation between the two means in the tangent space, and then weighting by the information of the two estimates. The information (inverse covariance) adds, as usual:

$$\begin{aligned}\boldsymbol{\Sigma}_c &= \left(\boldsymbol{\Sigma}_0^{-1} + \boldsymbol{\Sigma}_1^{-1}\right)^{-1} &(48)\\ &= \boldsymbol{\Sigma}_0 - \boldsymbol{\Sigma}_0 \left(\boldsymbol{\Sigma}_0 + \boldsymbol{\Sigma}_1\right)^{-1} \boldsymbol{\Sigma}_0 &(49)\\ \mathbf{v} &\equiv \mathbf{R}_1 \ominus \mathbf{R}_0 &(50)\\ &= \ln\left(\mathbf{R}_1 \cdot \mathbf{R}_0^{-1}\right) &(51)\\ \mathbf{R}_c &= \exp\left(\boldsymbol{\Sigma}_c \cdot \boldsymbol{\Sigma}_1^{-1} \cdot \mathbf{v}\right) \cdot \mathbf{R}_0 &(52)\end{aligned}$$

2.6 Extended Kalman Filtering in SO(3)

Equation 47 could be used as the dynamics update in an extended Kalman filter (EKF), where $(\mathbf{R}_0, \boldsymbol{\Sigma}_0)$ is the prior state and $(\mathbf{R}_1, \boldsymbol{\Sigma}_1)$ is the dynamic model.

Note that Equation 49 is actually the EKF measurement update for the covariance and Equation 52 is the measurement update for the mean, assuming a trivial measurement Jacobian (identity matrix). The tangent vector $\mathbf{v}$ is the innovation.

In this case of trivial measurement Jacobian, the Kalman gain $\mathbf{K}$ is defined

$$\mathbf{K} \equiv \boldsymbol{\Sigma}_0 \left(\boldsymbol{\Sigma}_0 + \boldsymbol{\Sigma}_1\right)^{-1} \tag{53}$$

so that the Kalman update can be written in its standard form:

$$\begin{aligned}\mathbf{R}_c &= \mathbf{R}_0 \oplus (\mathbf{K} \cdot \mathbf{v}) &(54)\\ &= \exp(\mathbf{K} \cdot \mathbf{v}) \cdot \mathbf{R}_0 &(55)\\ \boldsymbol{\Sigma}_c &= (\mathbf{I} - \mathbf{K}) \cdot \boldsymbol{\Sigma}_0 &(56)\end{aligned}$$

Labelling the above in the standard EKF framework, the state covariance is given by $\mathbf{\Sigma}_0$ and the measurement noise is given by $\mathbf{\Sigma}_1$. Note that Eq. 55 is mathematically identical to Eq. 52, and Eq. 56 is identical to Eq. 49.

The case of non-trivial measurement or dynamics Jacobians is a simple modification of the equations given here.

3 SE(3): Rigid transformations in 3D space

3.1 Representation

The group of rigid transformations in 3D space, SE(3), is well represented by linear transformations on homogeneous four-vectors:

$$\begin{aligned} \mathbf{R} &\in \mathrm{SO}(3),\ \mathbf{t} \in \mathbb{R}^3 & (57) \\ C &= \left(\begin{array}{c|c} \mathbf{R} & \mathbf{t} \\ \hline \mathbf{0} & 1 \end{array}\right) \in \mathrm{SE}(3) & (58) \end{aligned}$$

Note that, in an implementation, only $\mathbf{R}$ and $\mathbf{t}$ need to be stored. The remaining matrix structure can be implicitly imposed.

This representation, as in SO(3), means that transformation composition and inversion are coincident with matrix multiplication and inversion:

$$\begin{aligned} C_1, C_2 &\in \mathrm{SE}(3) & (59) \\ C_1 \cdot C_2 &= \left(\begin{array}{c|c} \mathbf{R}_1 & \mathbf{t}_1 \\ \hline \mathbf{0} & 1 \end{array}\right) \cdot \left(\begin{array}{c|c} \mathbf{R}_2 & \mathbf{t}_2 \\ \hline \mathbf{0} & 1 \end{array}\right) & (60) \\ &= \left(\begin{array}{c|c} \mathbf{R}_1\mathbf{R}_2 & \mathbf{R}_1\mathbf{t}_2 + \mathbf{t}_1 \\ \hline \mathbf{0} & 1 \end{array}\right) & (61) \\ C_1^{-1} &= \left(\begin{array}{c|c} \mathbf{R}_1^T & -\mathbf{R}_1^T\mathbf{t} \\ \hline \mathbf{0} & 1 \end{array}\right) & (62) \end{aligned}$$

The matrix representation also makes the group action on 3D points and vectors clear:

$$\begin{aligned} \mathbf{x} &= \begin{pmatrix} x & y & z & w \end{pmatrix}^T \in \mathbb{RP}^3 \quad (\lambda\mathbf{x} \simeq \mathbf{x}\,\forall\lambda \in \mathbb{R}) \\ C \cdot \mathbf{x} &= \left(\begin{array}{c|c} \mathbf{R} & \mathbf{t} \\ \hline \mathbf{0} & 1 \end{array}\right) \cdot \mathbf{x} & (63) \\ &= \begin{pmatrix} \mathbf{R}\begin{pmatrix} x & y & z \end{pmatrix}^T + w\mathbf{t} \\ w \end{pmatrix} & (64) \end{aligned}$$

Typically, $w = 1$, so that $\mathbf{x}$ is a Cartesian point. The action by matrix-vector multiplication corresponds to first rotating $\mathbf{x}$ and then translating it. For direction vectors, encoded with $w = 0$, translation is ignored.

The Lie algebra se(3) is the set of 4×4 matrices corresponding to differential translations and rotations (as in so(3)). There are thus six generators of the algebra:

$$
\begin{array}{c}
G_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad G_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
G_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad G_5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad G_6 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{array} \tag{65}
$$

An element of se(3) is then represented by multiples of the generators:

$$
\begin{pmatrix} \mathbf{u} & \boldsymbol{\omega} \end{pmatrix}^T \in \mathbb{R}^6 \tag{66}
$$

$$
\mathbf{u}_1 G_1 + \mathbf{u}_2 G_2 + \mathbf{u}_3 G_3 + \boldsymbol{\omega}_1 G_4 + \boldsymbol{\omega}_2 G_5 + \boldsymbol{\omega}_3 G_6 \in \mathrm{se}(3) \tag{67}
$$

For convenience, we write $\begin{pmatrix} \mathbf{u} & \boldsymbol{\omega} \end{pmatrix}^T \in \mathrm{se}(3)$, with multiplication against the generators implied.

3.2 Exponential Map

The exponential map from se(3) to SE(3) is the matrix exponential on a linear combination of the generators:

$$
\boldsymbol{\delta} = \begin{pmatrix} \mathbf{u} & \boldsymbol{\omega} \end{pmatrix} \in \mathrm{se}(3) \tag{68}
$$

$$
\exp(\boldsymbol{\delta}) = \exp\left(\begin{array}{c|c} \boldsymbol{\omega}_\times & \mathbf{u} \\ \hline \mathbf{0} & 0 \end{array}\right) \tag{69}
$$

$$
= \mathbf{I} + \left(\begin{array}{c|c} \boldsymbol{\omega}_\times & \mathbf{u} \\ \hline \mathbf{0} & 0 \end{array}\right) + \frac{1}{2!}\left(\begin{array}{c|c} \boldsymbol{\omega}_\times^2 & \boldsymbol{\omega}_\times \mathbf{u} \\ \hline \mathbf{0} & 0 \end{array}\right) + \frac{1}{3!}\left(\begin{array}{c|c} \boldsymbol{\omega}_\times^3 & \boldsymbol{\omega}_\times^2 \mathbf{u} \\ \hline \mathbf{0} & 0 \end{array}\right) + \cdots \tag{70}
$$

The rotation block is the same as for SO(3), but the translation component is a different power series:

$$
\exp\left(\begin{array}{c|c} \boldsymbol{\omega}_\times & \mathbf{u} \\ \hline \mathbf{0} & 0 \end{array}\right) = \left(\begin{array}{c|c} \exp(\boldsymbol{\omega}_\times) & \mathbf{V}\mathbf{u} \\ \hline \mathbf{0} & 1 \end{array}\right) \tag{71}
$$

$$
\mathbf{V} = \mathbf{I} + \frac{1}{2!}\boldsymbol{\omega}_\times + \frac{1}{3!}\boldsymbol{\omega}_\times^2 + \cdots \tag{72}
$$

Again using the identity from Eq. 9, we split the terms by odd and even powers, and factor out :

$$
\begin{aligned}
\mathbf{V} &= \mathbf{I} + \sum_{i=0}^{\infty}\left[\frac{\boldsymbol{\omega}_\times^{2i+1}}{(2i+2)!} + \frac{\boldsymbol{\omega}_\times^{2i+2}}{(2i+3)!}\right] && (73)\\
&= \mathbf{I} + \left(\sum_{i=0}^{\infty}\frac{(-1)^i\theta^{2i}}{(2i+2)!}\right)\boldsymbol{\omega}_\times + \left(\sum_{i=0}^{\infty}\frac{(-1)^i\theta^{2i}}{(2i+3)!}\right)\boldsymbol{\omega}_\times^2 && (74)
\end{aligned}
$$

The coefficients can be identified with Taylor expansions, yielding a formula for $\mathbf{V}$:

$$
\begin{aligned}
\mathbf{V} &= \mathbf{I} + \left(\frac{1}{2!} - \frac{\theta^2}{4!} + \frac{\theta^4}{6!} + \cdots\right)\boldsymbol{\omega}_\times + \left(\frac{1}{3!} - \frac{\theta^2}{5!} + \frac{\theta^4}{7!} + \cdots\right)\boldsymbol{\omega}_\times^2 && (75)\\
&= \mathbf{I} + \left(\frac{1-\cos\theta}{\theta^2}\right)\boldsymbol{\omega}_\times + \left(\frac{\theta - \sin\theta}{\theta^3}\right)\boldsymbol{\omega}_\times^2 && (76)
\end{aligned}
$$

Thus the exponential map has a closed-form representation:

$$
\begin{aligned}
\mathbf{u}, \boldsymbol{\omega} &\in \mathbb{R}^3 && (77)\\
\theta &= \sqrt{\boldsymbol{\omega}^T\boldsymbol{\omega}} && (78)\\
A &= \frac{\sin\theta}{\theta} && (79)\\
B &= \frac{1-\cos\theta}{\theta^2} && (80)\\
C &= \frac{1-A}{\theta^2} && (81)\\
\mathbf{R} &= \mathbf{I} + A\boldsymbol{\omega}_\times + B\boldsymbol{\omega}_\times^2 && (82)\\
\mathbf{V} &= \mathbf{I} + B\boldsymbol{\omega}_\times + C\boldsymbol{\omega}_\times^2 && (83)\\
\exp\begin{pmatrix}\mathbf{u}\\ \boldsymbol{\omega}\end{pmatrix} &= \left(\begin{array}{c|c}\mathbf{R} & \mathbf{V}\mathbf{u}\\ \hline \mathbf{0} & 1\end{array}\right) && (84)
\end{aligned}
$$

18

For implementation purposes, Taylor expansions of A, B, and C should be used when θ^2 is small.

The matrix $\mathbf{V}$ has a closed-form inverse:

$$
\mathbf{V}^{-1} = \mathbf{I} - \frac{1}{2}\boldsymbol{\omega}_\times + \frac{1}{\theta^2}\left(1 - \frac{A}{2B}\right)\boldsymbol{\omega}_\times^2 \tag{85}
$$

The ln() function on SE(3) can be implemented by first finding $\ln(\mathbf{R})$ as shown in Eq. 18, then computing $\mathbf{u} = \mathbf{V}^{-1} \cdot \mathbf{t}$.

3.3 Adjoint

The adjoint in SE(3) is computed from the generators, just as in SO(3):

$$\begin{aligned}
\boldsymbol{\delta} = \begin{pmatrix} \mathbf{u} & \boldsymbol{\omega} \end{pmatrix}^T \quad &\in \quad \mathrm{se}(3),\ C = \left(\begin{array}{c|c} \mathbf{R} & \mathbf{t} \\ \hline \mathbf{0} & 1 \end{array}\right) \in \mathrm{SE}(3) && (86) \\
C \cdot \exp(\boldsymbol{\delta}) \quad &= \quad \exp(\mathrm{Adj}_C \cdot \boldsymbol{\delta}) \cdot C \\
\exp(\mathrm{Adj}_C \cdot \boldsymbol{\delta}) \quad &= \quad C \cdot \exp(\boldsymbol{\delta}) \cdot C^{-1} && (87) \\
\mathrm{Adj}_C \cdot \boldsymbol{\delta} \quad &= \quad C \cdot \left(\sum_{i=1}^{6} \boldsymbol{\delta}_i G_i \right) \cdot C^{-1} && (88) \\
&= \quad \begin{pmatrix} \mathbf{R}\mathbf{u} + \mathbf{t} \times \mathbf{R}\boldsymbol{\omega} \\ \mathbf{R}\boldsymbol{\omega} \end{pmatrix} && (89) \\
\Longrightarrow \ \mathrm{Adj}_C \quad &= \quad \left(\begin{array}{c|c} \mathbf{R} & \mathbf{t}_\times \mathbf{R} \\ \hline \mathbf{0} & \mathbf{R} \end{array}\right) \in \mathbb{R}^{6\times 6} && (90)
\end{aligned}$$

Note that moving a tangent vector via the adjoint mixes the rotation component into the translation component.

3.4 Jacobians

Consider $C = \left(\begin{array}{c|c} \mathbf{R} & \mathbf{t} \\ \hline \mathbf{0} & 1 \end{array}\right) \in \mathrm{SE}(3)$ and $\mathbf{x} \in \mathcal{R}^3$. The transformation of vector $\mathbf{x}$ by C is given by multiplication:

$$\begin{aligned}
\mathbf{y} \quad &= \quad f(C, \mathbf{x}) = \left(\begin{array}{c|c} \mathbf{R} & \mathbf{t} \end{array} \right) \cdot \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} && (91) \\
&= \quad \mathbf{R} \cdot \mathbf{x} + \mathbf{t} && (92)
\end{aligned}$$

Then differentiation by the vector is straightforward, as f is linear in $\mathbf{x}$:

$$\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \mathbf{R} \tag{93}$$

Just as with SO(3), differentiation by the transformation parameters is performed by left multiplying the product by the generators (here with their last rows removed):

$$\begin{aligned}
\frac{\partial \mathbf{y}}{\partial C} \quad &= \quad \left(\begin{array}{c|c|c} G_1 \mathbf{y} & \cdots & G_6 \mathbf{y} \end{array} \right) \\
&= \quad \left(\begin{array}{c|c} \mathbf{I} & -\mathbf{y}_\times \end{array} \right) && (94)
\end{aligned}$$

Again, differentiation of a product of transformations is trivial given the adjoint:

$$
\begin{aligned}
C &\equiv C_1 \cdot C_0 &(95)\\
\frac{\partial C}{\partial C_0} &= \frac{\partial}{\partial \boldsymbol{\delta}}\left[C_1 \cdot \exp(\boldsymbol{\delta}) \cdot C_0\right] &(96)\\
&= \mathrm{Adj}_{C_1} &(97)
\end{aligned}
$$

4 SO(2): Rotations in 2D space

Having treated SO(3), the 2D equivalent SO(3) is straightforward.

4.1 Representation

Elements of the rotation group in two dimensions, SO(2), are represented by 2D rotation matrices. Composition and inversion in the group correspond to matrix multiplication and inversion. Because rotation matrices are orthogonal, inversion is equivalent to transposition.

$$
\begin{aligned}
\mathbf{R} &\in \mathrm{SO}(2) &(98)\\
\mathbf{R}^{-1} &= \mathbf{R}^T &(99)
\end{aligned}
$$

The Lie algebra, so(2), is the set of 2×2 skew-symmetric matrices. The single generator of so(2) corresponds to the derivative of 2D rotation, evaluated at the identity:

$$
G = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (100)
$$

An element of so(2) is then any scalar multiple of the generator:

$$
\begin{aligned}
\theta &\in \mathbb{R} &(101)\\
\theta G &\in \mathrm{so}(2) &(102)
\end{aligned}
$$

We will simply write $\theta \in \mathrm{so}(2)$, and use $\theta_\times$ to represent the skew symmetric matrix θG.

4.2 Exponential Map

The exponential map that takes skew symmetric matrices to rotation matrices is simply the matrix exponential over a linear combination of the generators:

$$\begin{aligned}
\exp(\theta_\times) &\equiv \exp\begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} && (103)\\
&= \mathbf{I} + \theta_\times + \frac{1}{2!}\theta_\times^2 + \frac{1}{3!}\theta_\times^3 + \cdots && (104)\\
&= \mathbf{I} + \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} -\theta^2 & 0 \\ 0 & -\theta^2 \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} 0 & \theta^3 \\ -\theta^3 & 0 \end{pmatrix} && (105)
\end{aligned}$$

The resulting elements form the Taylor series expansion of $\sin\theta$ and $\cos\theta$:

$$\exp(\theta_\times) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SO}(2) \tag{106}$$

Thus the exponential map yields a rotation by θ radians.

The exponential map can be inverted, going from SO(2) to so(2):

$$\begin{aligned}
\mathbf{R} &\in \mathrm{SO}(2) && (107)\\
\ln(\mathbf{R}) = \theta &= \arctan(\mathbf{R}_{21}, \mathbf{R}_{11}) && (108)
\end{aligned}$$

4.3 Adjoint

Because rotations in the plane commute, the adjoint of SO(2) is the identity function.

5 SE(2): Rigid transformations in 2D space

The group SE(2) is the lower-dimensional analogue of SE(3). The group has three dimensions, corresponding to translation and rotation in the plane.

5.1 Representation

The group of rigid transformations in 2D space, SE(2), is represented by linear transformations on homogeneous three-vectors:

$$\begin{aligned}
\mathbf{R} &\in \mathrm{SO}(2),\ \mathbf{t} \in \mathbb{R}^2 && (109)\\
C &= \left(\begin{array}{c|c} \mathbf{R} & \mathbf{t} \\ \hline \mathbf{0} & 1 \end{array}\right) \in \mathrm{SE}(2) && (110)
\end{aligned}$$

Note that, in an implementation, only $\mathbf{R}$ and $\mathbf{t}$ need to be stored. The remaining matrix structure can remain implicit.

Transformation composition and inversion are coincident with matrix multiplication and inversion:

$$C_1, C_2 \quad \in \quad \mathrm{SE}(2) \tag{111}$$

$$C_1 \cdot C_2 \quad = \quad \left(\begin{array}{c|c} \mathbf{R}_1 & \mathbf{t}_1 \\ \hline \mathbf{0} & 1 \end{array}\right) \cdot \left(\begin{array}{c|c} \mathbf{R}_2 & \mathbf{t}_2 \\ \hline \mathbf{0} & 1 \end{array}\right) \tag{112}$$

$$= \quad \left(\begin{array}{c|c} \mathbf{R}_1\mathbf{R}_2 & \mathbf{R}_1\mathbf{t}_2 + \mathbf{t}_1 \\ \hline \mathbf{0} & 1 \end{array}\right) \tag{113}$$

$$C_1^{-1} \quad = \quad \left(\begin{array}{c|c} \mathbf{R}_1^T & -\mathbf{R}_1^T\mathbf{t} \\ \hline \mathbf{0} & 1 \end{array}\right) \tag{114}$$

The matrix representation also makes the group action on 2D points and vectors explicit:

$$\mathbf{x} \quad = \quad \left(\begin{array}{ccc} x & y & w \end{array} \right)^T \in \mathbb{RP}^2 \quad (\lambda\mathbf{x} \simeq \mathbf{x} \,\forall \lambda \in \mathbb{R})$$

$$C \cdot \mathbf{x} \quad = \quad \left(\begin{array}{c|c} \mathbf{R} & \mathbf{t} \\ \hline \mathbf{0} & 1 \end{array}\right) \cdot \mathbf{x} \tag{115}$$

$$= \quad \left(\begin{array}{c} \mathbf{R} \left(\begin{array}{cc} x & y \end{array} \right)^T + w\mathbf{t} \\ w \end{array} \right) \tag{116}$$

Typically, $w = 1$, so that $\mathbf{x}$ is a Cartesian point. The action by matrix-vector multiplication corresponds to first rotating $\mathbf{x}$ and then translating it. For direction vectors, encoded with $w = 0$, translation is ignored.

The Lie algebra se(2) is the set of 3×3 matrices corresponding to differential translations and rotation around the identity. There are thus three generators of the algebra:

$$G_1 = \left(\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), \quad G_2 = \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right), \quad G_3 = \left(\begin{array}{ccc} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \tag{117}$$

An element of se(2) is then represented by linear combinations of the generators:

$$\left(\begin{array}{ccc} \mathbf{u}_1 & \mathbf{u}_2 & \theta \end{array} \right)^T \quad \in \quad \mathbb{R}^3 \tag{118}$$

$$\mathbf{u}_1 G_1 + \mathbf{u}_2 G_2 + \theta G_3 \quad \in \quad \mathrm{se}(2) \tag{119}$$

For convenience, we write $\left(\begin{array}{cc} \mathbf{u} & \theta \end{array} \right)^T \in \mathrm{se}(2)$, with multiplication against the generators implied.

5.2 Exponential Map

As for all Lie groups in this document, the exponential map from se(2) to SE(2) is the matrix exponential on a linear combination of the generators:

$$\boldsymbol{\delta} = \begin{pmatrix} \mathbf{u} & \theta \end{pmatrix} \quad \in \quad \mathrm{se}(2) \tag{120}$$

$$\exp(\boldsymbol{\delta}) = \exp\left(\begin{array}{c|c} \theta_\times & \mathbf{u} \\ \hline \mathbf{0} & 0 \end{array}\right) \tag{121}$$

$$= \mathbf{I} + \left(\begin{array}{c|c} \theta_\times & \mathbf{u} \\ \hline \mathbf{0} & 0 \end{array}\right) + \frac{1}{2!}\left(\begin{array}{c|c} \theta_\times^2 & \theta_\times \mathbf{u} \\ \hline \mathbf{0} & 0 \end{array}\right) + \frac{1}{3!}\left(\begin{array}{c|c} \theta_\times^3 & \theta_\times^2 \mathbf{u} \\ \hline \mathbf{0} & 0 \end{array}\right) + \cdots \tag{122}$$

The rotation block is the same as for SO(2), but the translation component is a different power series:

$$\exp\left(\begin{array}{c|c} \theta_\times & \mathbf{u} \\ \hline \mathbf{0} & 0 \end{array}\right) = \left(\begin{array}{c|c} \exp(\theta_\times) & \mathbf{V}\mathbf{u} \\ \hline \mathbf{0} & 1 \end{array}\right) \tag{123}$$

$$\mathbf{V} = \mathbf{I} + \frac{1}{2!}\theta_\times + \frac{1}{3!}\theta_\times^2 + \cdots \tag{124}$$

We split the terms by odd and even powers:

$$\mathbf{V} = \sum_{i=0}^{\infty}\left[\frac{\theta_\times^{2i}}{(2i+1)!} + \frac{\theta_\times^{2i+1}}{(2i+2)!}\right] \tag{125}$$

Two identities (easily confirmed by induction) are useful for collapsing the series:

$$\theta_\times^{2i} = (-1)^i \theta^{2i} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{126}$$

$$\theta_\times^{2i+1} = (-1)^i \theta^{2i+1} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{127}$$

Direct application of the identies yields a reduced expression for $\mathbf{V}$ in terms of diagonal and skew-symmetric components:

$$\mathbf{V} = \sum_{i=0}^{\infty}(-1)^i\theta^{2i}\left[\frac{1}{(2i+1)!}\cdot\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\theta}{(2i+2)!}\cdot\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right] \tag{128}$$

$$= \left(\sum_{i=0}^{\infty}\frac{(-1)^i\theta^{2i}}{(2i+1)!}\right)\cdot\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \left(\sum_{i=0}^{\infty}\frac{(-1)^i\theta^{2i+1}}{(2i+2)!}\right)\cdot\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{129}$$

The coefficients can be identified with Taylor expansions:

$$
\begin{aligned}
\mathbf{V} &= \left(1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} + \cdots\right) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \left(\frac{\theta}{2!} - \frac{\theta^3}{4!} + \frac{\theta^5}{6!} + \cdots\right) \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} && (130) \\
&= \left(\frac{\sin\theta}{\theta}\right) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \left(\frac{1-\cos\theta}{\theta}\right) \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} && (131) \\
&= \frac{1}{\theta} \cdot \begin{pmatrix} \sin\theta & -(1-\cos\theta) \\ 1-\cos\theta & \sin\theta \end{pmatrix} && (132)
\end{aligned}
$$

For implementation purposes, Taylor expansions should be used for $\mathbf{V}$ when θ is small.

The ln() function on SE(2) can be implemented by first recovering $\theta = \ln(\mathbf{R})$ as shown in Eq. 108, then solving $\mathbf{V}\mathbf{u} = \mathbf{t}$ for $\mathbf{u}$ in closed form:

$$
\begin{aligned}
A &\equiv \frac{\sin\theta}{\theta} && (133) \\
B &\equiv \frac{1-\cos\theta}{\theta} && (134) \\
\mathbf{V}^{-1} &= \frac{1}{A^2+B^2} \begin{pmatrix} A & B \\ -B & A \end{pmatrix} && (135) \\
\ln\left(\begin{array}{c|c} \mathbf{R} & \mathbf{t} \\ \hline \mathbf{0} & 1 \end{array}\right) &= \begin{pmatrix} \mathbf{V}^{-1} \cdot \mathbf{t} \\ \theta \end{pmatrix} \in \mathrm{se}(2) && (136)
\end{aligned}
$$

5.3 Adjoint

The adjoint in SE(2) is computed from the generators:

$$
\begin{aligned}
\boldsymbol{\delta} = \begin{pmatrix} \mathbf{u} & \theta \end{pmatrix}^T &\in \mathrm{se}(2),\ C = \left(\begin{array}{c|c} \mathbf{R} & \mathbf{t} \\ \hline \mathbf{0} & 1 \end{array}\right) \in \mathrm{SE}(2) && (137) \\
\mathrm{Adj}_C \cdot \boldsymbol{\delta} &= C \cdot \left(\sum_{i=1}^{3} \boldsymbol{\delta}_i G_i\right) \cdot C^{-1} && (138) \\
&= \begin{pmatrix} \mathbf{R}\mathbf{u} + \theta \begin{pmatrix} \mathbf{t}_2 \\ -\mathbf{t}_1 \end{pmatrix} \\ \theta \end{pmatrix} && (139) \\
\implies \mathrm{Adj}_C &= \left(\begin{array}{c|c} \mathbf{R} & \begin{matrix} \mathbf{t}_2 \\ -\mathbf{t}_1 \end{matrix} \\ \hline \mathbf{0} & 1 \end{array}\right) \in \mathbb{R}^{3\times 3} && (140)
\end{aligned}
$$

Note that moving a tangent vector via the adjoint mixes the rotation component into the translation component.

6 Sim(3): Similarity Transformations in 3D space

6.1 Representation

Similarity transformations are combinations of rigid transformation and scaling. The group of similarity transforms in 3D space, Sim(3), has a nearly identical representation to SE(3), with an additional scale factor:

$$
\begin{aligned}
\mathbf{R} &\in \mathrm{SO}(3),\ \mathbf{t} \in \mathbb{R}^3,\ s \in \mathbb{R} && (141)\\
T &= \left(\begin{array}{c|c} \mathbf{R} & \mathbf{t} \\ \hline \mathbf{0} & s^{-1} \end{array}\right) \in \mathrm{Sim}(3) && (142)
\end{aligned}
$$

Again, group operations map are isomorphic with matrix operations:

$$
\begin{aligned}
T_1, T_2 &\in \mathrm{Sim}(3) && (143)\\
T_1 \cdot T_2 &= \left(\begin{array}{c|c} \mathbf{R}_1 & \mathbf{t}_1 \\ \hline \mathbf{0} & s_1^{-1} \end{array}\right) \cdot \left(\begin{array}{c|c} \mathbf{R}_2 & \mathbf{t}_2 \\ \hline \mathbf{0} & s_2^{-1} \end{array}\right) && (144)\\
&= \left(\begin{array}{c|c} \mathbf{R}_1\mathbf{R}_2 & \mathbf{R}_1\mathbf{t}_2 + s_2^{-1}\mathbf{t}_1 \\ \hline \mathbf{0} & (s_1 \cdot s_2)^{-1} \end{array}\right) && (145)\\
T_1^{-1} &= \left(\begin{array}{c|c} \mathbf{R}_1^T & -s_1\mathbf{R}_1^T\mathbf{t} \\ \hline \mathbf{0} & s_1 \end{array}\right) && (146)
\end{aligned}
$$

The group action on 3D points also encodes scaling by s:

$$
\begin{aligned}
\mathbf{x} &= \begin{pmatrix} x & y & z & w \end{pmatrix}^T \in \mathbb{RP}^3 \quad (\lambda\mathbf{x} \simeq \mathbf{x}\,\forall\lambda \in \mathbb{R}) \\
T \cdot \mathbf{x} &= \left(\begin{array}{c|c} \mathbf{R} & \mathbf{t} \\ \hline \mathbf{0} & s^{-1} \end{array}\right) \cdot \mathbf{x} && (147)\\
&= \begin{pmatrix} \mathbf{R}\begin{pmatrix} x & y & z \end{pmatrix}^T + w\mathbf{t} \\ s^{-1}w \end{pmatrix} && (148)\\
&\simeq \begin{pmatrix} s\left(\mathbf{R}\begin{pmatrix} x & y & z \end{pmatrix}^T + w\mathbf{t}\right) \\ w \end{pmatrix} && (149)
\end{aligned}
$$

In the typical case with $w = 1$, this corresponds to rigid transformation followed by scaling.

The generators of the Lie algebra sim(3) are identical to those of se(3) (Eq. 65), with the addition of a generator corresponding to scale change:

$$
G_7 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad (150)
$$

An element of sim(3) is represented by multiples of the generators:

$$\begin{pmatrix} \mathbf{u} & \boldsymbol{\omega} & \lambda \end{pmatrix}^T \in \mathbb{R}^7 \tag{151}$$

$$\mathbf{u}_1 G_1 + \mathbf{u}_2 G_2 + \mathbf{u}_3 G_3 + \boldsymbol{\omega}_1 G_4 + \boldsymbol{\omega}_2 G_5 + \boldsymbol{\omega}_3 G_6 + \lambda G_7 \in \mathrm{sim}(3) \tag{152}$$

For convenience, we write $\begin{pmatrix} \mathbf{u} & \boldsymbol{\omega} & \lambda \end{pmatrix}^T \in \mathrm{sim}(3)$, with multiplication against the generators implied.

6.2 Exponential Map

As above, the exponential map from sim(3) to Sim(3) is the matrix exponential on a linear combination of the generators:

$$\boldsymbol{\delta} = \begin{pmatrix} \mathbf{u} & \boldsymbol{\omega} & \lambda \end{pmatrix} \in \mathrm{sim}(3) \tag{153}$$

$$\exp(\boldsymbol{\delta}) = \exp\left(\begin{array}{c|c} \boldsymbol{\omega}_\times & \mathbf{u} \\ \hline \mathbf{0} & -\lambda \end{array}\right) \tag{154}$$

$$= \mathbf{I} + \left(\begin{array}{c|c} \boldsymbol{\omega}_\times & \mathbf{u} \\ \hline \mathbf{0} & \lambda \end{array}\right) + \frac{1}{2!}\left(\begin{array}{c|c} \boldsymbol{\omega}_\times^2 & \boldsymbol{\omega}_\times \mathbf{u} - \lambda \mathbf{u} \\ \hline \mathbf{0} & \lambda^2 \end{array}\right) + \frac{1}{3!}\left(\begin{array}{c|c} \boldsymbol{\omega}_\times^3 & \boldsymbol{\omega}_\times^2 \mathbf{u} - \lambda \boldsymbol{\omega}_\times \mathbf{u} + \lambda^2 \mathbf{u} \\ \hline \mathbf{0} & -\lambda^3 \end{array}\right) + \ldots \tag{155}$$

The series is similar to that for se(3), but now rotation, translation and scale are being interleaved infinitesimally. The rotation and scale components of the exponential are immediately clear, but the translation component involves the mixing of the three. We can write out the series for the translation multiplier:

$$\exp\left(\begin{array}{c|c} \boldsymbol{\omega}_\times & \mathbf{u} \\ \hline \mathbf{0} & 0 \end{array}\right) = \left(\begin{array}{c|c} \exp(\boldsymbol{\omega}_\times) & \mathbf{V}\mathbf{u} \\ \hline \mathbf{0} & \exp(-\lambda) \end{array}\right) \tag{156}$$

$$\mathbf{V} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{\boldsymbol{\omega}_\times^{n-k} (-\lambda)^k}{(n+1)!} \tag{157}$$

$$= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \frac{\boldsymbol{\omega}_\times^{n-k} (-\lambda)^k}{(n+1)!} \tag{158}$$

$$= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{\boldsymbol{\omega}_\times^{j} (-\lambda)^k}{(j+k+1)!} \tag{159}$$

Again letting $\theta^2 = \boldsymbol{\omega}^T \boldsymbol{\omega}$, and using the identity from Eq. 9, we partition the terms into odd and even powers of $\boldsymbol{\omega}_\times$, and factor:

$$\mathbf{V} = \left(\sum_{k=0}^{\infty} \frac{(-\lambda)^k}{(k+1)!}\right)\mathbf{I} + \sum_{k=0}^{\infty} (-\lambda)^k \sum_{i=0}^{\infty} \left[\frac{\boldsymbol{\omega}_\times^{2i+1}}{(2i+k+2)!} + \frac{\boldsymbol{\omega}_\times^{2i+2}}{(2i+k+3)!}\right] \tag{160}$$

$$= \left(\sum_{k=0}^{\infty} \frac{(-\lambda)^k}{(k+1)!}\right)\mathbf{I} + \left(\sum_{k=0}^{\infty}\sum_{i=0}^{\infty} \frac{(-1)^i \theta^{2i} (-\lambda)^k}{(2i+k+2)!}\right)\boldsymbol{\omega}_\times + \left(\sum_{k=0}^{\infty}\sum_{i=0}^{\infty} \frac{(-1)^i \theta^{2i} (-\lambda)^k}{(2i+k+3)!}\right)\boldsymbol{\omega}_\times^2 \tag{161}$$

The first coefficient is easily identified as a Taylor series, leaving the other two coefficients to be analyzed:

$$
\begin{aligned}
\mathbf{V} &= A\mathbf{I} + B\boldsymbol{\omega}_\times + C\boldsymbol{\omega}_\times^2 && (162)\\
A &= \frac{1-\exp(-\lambda)}{\lambda} && (163)\\
B &= \sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\frac{(-1)^i\theta^{2i}(-\lambda)^k}{(2i+k+2)!} && (164)\\
C &= \sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\frac{(-1)^i\theta^{2i}(-\lambda)^k}{(2i+k+3)!} && (165)
\end{aligned}
$$

Consider coefficient B:

$$
\begin{aligned}
B &= \sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\frac{(-1)^i\theta^{2i}(-\lambda)^k}{(2i+k+2)!} && (166)\\
&= \sum_{i=0}^{\infty}\left[(-1)^i\theta^{2i}\sum_{k=0}^{\infty}\frac{(-\lambda)^k}{(2i+k+2)!}\right] && (167)\\
&= \sum_{i=0}^{\infty}\left[\frac{(-1)^i\theta^{2i}}{\lambda^{2i}}\sum_{k=0}^{\infty}\frac{(-\lambda)^{2i+k}}{(2i+k+2)!}\right] && (168)\\
&= \sum_{i=0}^{\infty}\left[\frac{(-1)^i\theta^{2i}}{\lambda^{2i}}\sum_{m=2i}^{\infty}\frac{(-\lambda)^m}{(m+2)!}\right] && (169)\\
&= \sum_{i=0}^{\infty}\left(\frac{(-1)^i\theta^{2i}}{\lambda^{2i}}\left[\sum_{m=0}^{\infty}\frac{(-\lambda)^m}{(m+2)!}-\sum_{m=0}^{2i-1}\frac{(-\lambda)^m}{(m+2)!}\right]\right) && (170)\\
&= \sum_{i=0}^{\infty}\left(\frac{(-1)^i\theta^{2i}}{\lambda^{2i}}\left[\sum_{m=0}^{\infty}\frac{(-\lambda)^m}{(m+2)!}\right]\right)-\sum_{i=0}^{\infty}\left(\frac{(-1)^i\theta^{2i}}{\lambda^{2i}}\sum_{m=0}^{2i-1}\frac{(-\lambda)^m}{(m+2)!}\right) && (171)
\end{aligned}
$$

Manipulating the second term yields a more useful form:

$$
\begin{aligned}
L &\equiv \sum_{i=0}^{\infty}\left(\frac{(-1)^i\theta^{2i}}{\lambda^{2i}}\left[\sum_{m=0}^{\infty}\frac{(-\lambda)^m}{(m+2)!}\right]\right) && (172)\\
B &= L-\sum_{i=0}^{\infty}\left(\frac{(-1)^i\theta^{2i}}{\lambda^{2i}}\sum_{m=0}^{2i-1}\frac{(-\lambda)^m}{(m+2)!}\right) && (173)\\
&= L-\sum_{i=0}^{\infty}\left(\frac{(-1)^i\theta^{2i}}{\lambda^{2i}}\sum_{p=0}^{i-1}\left[\frac{(-\lambda)^{2p}}{(2p+2)!}+\frac{(-\lambda)^{2p+1}}{(2p+3)!}\right]\right) && (174)\\
&= L-\sum_{i=0}^{\infty}\left(\frac{(-1)^i\theta^{2i}}{\lambda^{2i}}\sum_{p=0}^{i-1}(-\lambda)^{2p}\left[\frac{1}{(2p+2)!}-\frac{\lambda}{(2p+3)!}\right]\right) && (175)\\
&= L-\sum_{p<i}^{\infty}\left(\frac{(-1)^i\theta^{2i}}{\lambda^{2i}}(-\lambda)^{2p}\left[\frac{1}{(2p+2)!}-\frac{\lambda}{(2p+3)!}\right]\right) && (176)\\
&= L-\sum_{p<i}^{\infty}\left(\frac{(-1)^i\theta^{2i}}{\lambda^{2(i-p)}}\left[\frac{1}{(2p+2)!}-\frac{\lambda}{(2p+3)!}\right]\right) && (177)\\
&= L-\sum_{p=0}^{\infty}\sum_{q=1}^{\infty}\left(\frac{(-1)^q\theta^{2q}}{\lambda^{2q}}\left[\frac{(-1)^p\theta^{2p}}{(2p+2)!}-\lambda\left(\frac{(-1)^p\theta^{2p}}{(2p+3)!}\right)\right]\right) && (178)\\
&= L-\left(\sum_{q=1}^{\infty}\frac{(-1)^q\theta^{2q}}{\lambda^{2q}}\right)\left(\sum_{p=0}^{\infty}\left[\frac{(-1)^p\theta^{2p}}{(2p+2)!}-\lambda\left(\frac{(-1)^p\theta^{2p}}{(2p+3)!}\right)\right]\right) && (179)\\
&= \left(\textstyle\sum_{i=0}^{\infty}\frac{(-1)^i\theta^{2i}}{\lambda^{2i}}\right)\left(\textstyle\sum_{m=0}^{\infty}\frac{(-\lambda)^m}{(m+2)!}\right) && (180)\\
&\quad -\left(\textstyle\sum_{q=0}^{\infty}\frac{(-1)^q\theta^{2q}}{\lambda^{2q}}-1\right)\left(\textstyle\sum_{p=0}^{\infty}\left[\frac{(-1)^p\theta^{2p}}{(2p+2)!}-\lambda\left(\frac{(-1)^p\theta^{2p}}{(2p+3)!}\right)\right]\right)
\end{aligned}
$$

Relabel the factors:

$$
\begin{aligned}
B &= \alpha\cdot\beta-(\alpha-1)\cdot\gamma && (181)\\
&= \alpha\cdot(\beta-\gamma)+\gamma && (182)\\
\alpha &\equiv \sum_{i=0}^{\infty}\frac{(-1)^i\theta^{2i}}{\lambda^{2i}} && (183)\\
\beta &\equiv \sum_{m=0}^{\infty}\frac{(-\lambda)^m}{(m+2)!} && (184)\\
\gamma &\equiv \sum_{p=0}^{\infty}\left[\frac{(-1)^p\theta^{2p}}{(2p+2)!}-\lambda\left(\frac{(-1)^p\theta^{2p}}{(2p+3)!}\right)\right] && (185)
\end{aligned}
$$

The factors are then recognised as Taylor series:

$$\alpha = \frac{\lambda^2}{\lambda^2 + \theta^2} \tag{186}$$

$$\beta = \frac{\exp(-\lambda) - 1 + \lambda}{\lambda^2} \tag{187}$$

$$\gamma = \frac{1 - \cos\theta}{\theta^2} - \lambda \left(\frac{\theta - \sin\theta}{\theta^3} \right) \tag{188}$$

By similar algebra, a formula for coefficient C can be derived from Eq. 165:

$$C = \alpha \cdot (\mu - \nu) + \nu \tag{189}$$

$$\mu = \frac{1 - \lambda + \frac{1}{2}\lambda^2 - \exp(-\lambda)}{\lambda^2} \tag{190}$$

$$\nu = \frac{\theta - \sin\theta}{\theta^3} - \lambda \left(\frac{\cos\theta - 1 + \frac{\theta^2}{2}}{\theta^4} \right) \tag{191}$$

Combining these results gives a closed-form exponential map for sim(3):

$$
\begin{aligned}
\begin{pmatrix} \mathbf{u} & \boldsymbol{\omega} & \lambda \end{pmatrix}^T &\in \mathrm{sim}(3) && (192)\\
\theta^2 &= \boldsymbol{\omega}^T\boldsymbol{\omega} && (193)\\
X &= \frac{\sin\theta}{\theta} && (194)\\
Y &= \frac{1-\cos\theta}{\theta^2} && (195)\\
Z &= \frac{1-X}{\theta^2} && (196)\\
W &= \frac{\frac{1}{2}-Y}{\theta^2} && (197)\\
\alpha &= \frac{\lambda^2}{\lambda^2+\theta^2} && (198)\\
\beta &= \frac{\exp(-\lambda)-1+\lambda}{\lambda^2} && (199)\\
\gamma &= Y-\lambda Z && (200)\\
\mu &= \frac{1-\lambda+\frac{1}{2}\lambda^2-\exp(-\lambda)}{\lambda^2} && (201)\\
\nu &= Z-\lambda W && (202)\\
A &= \frac{1-\exp(-\lambda)}{\lambda} && (203)\\
B &= \alpha\cdot(\beta-\gamma)+\gamma && (204)\\
C &= \alpha\cdot(\mu-\nu)+\nu && (205)\\
\mathbf{R} &= \mathbf{I}+a\boldsymbol{\omega}_\times+b\boldsymbol{\omega}_\times^2 && (206)\\
\mathbf{V} &= A\mathbf{I}+B\boldsymbol{\omega}_\times+C\boldsymbol{\omega}_\times^2 && (207)\\
\exp\begin{pmatrix}\mathbf{u}\\ \boldsymbol{\omega}\\ \lambda\end{pmatrix} &= \left(\begin{array}{c|c}\mathbf{R} & \mathbf{Vu}\\ \hline \mathbf{0} & \exp(-\lambda)\end{array}\right) && (208)
\end{aligned}
$$

Again, Taylor expansions should be used when λ^2 or θ^2 is small. The ln() function can be implemented by first recovering $\boldsymbol{\omega}$ and λ, constructing $\mathbf{V}$, and then solving for $\mathbf{u}$ (as in the SE(3) case).

6.3 Adjoint

The adjoint is computed from a linear combination of the generators:

$$
\begin{aligned}
\boldsymbol{\delta} = \begin{pmatrix} \mathbf{u} & \boldsymbol{\omega} & \lambda \end{pmatrix}^T &\in \mathrm{sim}(3),\ T = \left(\begin{array}{c|c} \mathbf{R} & \mathbf{t} \\ \hline \mathbf{0} & s^{-1} \end{array}\right) \in \mathrm{Sim}(3) && (209) \\
T \cdot \exp(\boldsymbol{\delta}) &= \exp(\mathrm{Adj}_T \cdot \boldsymbol{\delta}) \cdot T \\
\exp(\mathrm{Adj}_T \cdot \boldsymbol{\delta}) &= T \cdot \exp(\boldsymbol{\delta}) \cdot T^{-1} && (210) \\
\mathrm{Adj}_T \cdot \boldsymbol{\delta} &= T \cdot \left(\sum_{i=1}^{7} \boldsymbol{\delta}_i G_i \right) \cdot T^{-1} && (211) \\
&= \begin{pmatrix} s\left(\mathbf{R}\mathbf{u} + \mathbf{t} \times \mathbf{R}\boldsymbol{\omega} - s\mathbf{t}\right) \\ \mathbf{R}\boldsymbol{\omega} \\ -\lambda \end{pmatrix} && (212) \\
\implies \mathrm{Adj}_T &= \left(\begin{array}{c|c|c} s\mathbf{R} & s\mathbf{t}_\times \mathbf{R} & -s\mathbf{t} \\ \hline \mathbf{0} & \mathbf{R} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & 1 \end{array}\right) \in \mathbb{R}^{7\times 7} && (213)
\end{aligned}
$$

7 Interpolation

Consider a Lie group G, with two elements $a, b \in G$. We would like to *interpolate* between these elements, according to a parameter $t \in [0,1]$. Define a function that will perform the interpolation:

$$
\begin{aligned}
f : G \times G \times \mathbb{R} &\rightarrow G && (214) \\
f(a, b, 0) &= a && (215) \\
f(a, b, 1) &= b && (216)
\end{aligned}
$$

The function is defined by transforming the interpolation operation into the tangent space, performing the linear combination there, and then transforming the resulting tangent vector back onto the manifold. First consider the group element that takes a to b:

$$
\begin{aligned}
d &\equiv b \cdot a^{-1} \in G && (217) \\
d \cdot a &= b && (218)
\end{aligned}
$$

Now compute the corresponding Lie algebra vector and scale it in the tangent space:

$$
\mathbf{d}(t) = t \cdot \ln(d) \qquad (219)
$$

Then transform back into the manifold using the exponential map, yielding a "partial" transformation:

$$
d_t = \exp(\mathbf{d}(t)) \qquad (220)
$$

Combining these three steps gives a definition for f:

$$f(a,b,t) = d_t \cdot a \tag{221}$$
$$= \exp\left(t \cdot \ln\left(b \cdot a^{-1}\right)\right) \cdot a \tag{222}$$

Note that the result is always on the manifold, due to the properties of the exponential map. No projection or coercion is required. Furthermore, the linear transformation in the tangent space corresponds to moving along a *geodesic* of the manifold. So the interpolation always moves along the "shortest" transformation in the Lie group.

8 Uncertain Transformations

8.1 Sampling

Consider a Lie group G and its associated Lie algebra vector space $\mathfrak{g}$, with k degrees of freedom. We wish to represent Gaussian distributions over transformations in this group. Each such distribution has a mean transformation, $\mu \in G$, and a covariance matrix $\mathbf{\Sigma} \in \mathbb{R}^{k \times k}$. The algebra corresponds to tangent vectors around the identity element of the group. Thus, it is natural to express a sample x from the desired distribution in terms of a sample $\boldsymbol{\delta}$ drawn from a zero-mean Gaussian and the mean transformation μ:

$$\boldsymbol{\delta} \in \mathcal{N}\left(\mathbf{0}; \mathbf{\Sigma}\right) \tag{223}$$
$$x = \exp\left(\boldsymbol{\delta}\right) \cdot \mu \tag{224}$$

8.2 Transforming

This formulation allows convenient transformations of distributions through other group elements using the adjoint. Consider $x, y \in G$, and the distribution $(x, \mathbf{\Sigma})$ with mean x. Consider $y \cdot \tilde{x}$, where $\tilde{x}$ is a sample drawn from $(x, \mathbf{\Sigma})$:

$$\tilde{x} = \exp\left(\boldsymbol{\delta}\right) \cdot x \tag{225}$$
$$y \cdot \tilde{x} = y \cdot \exp\left(\boldsymbol{\delta}\right) \cdot x \tag{226}$$
$$= \exp\left(\mathrm{Adj}_y \boldsymbol{\delta}\right) \cdot y \cdot x \tag{227}$$

By the definition of covariance,

$$\mathbf{\Sigma} = \mathrm{E}\left[\boldsymbol{\delta} \cdot \boldsymbol{\delta}^T\right] \tag{228}$$

Given a linear transformation L, by linearity of expectation we have:

$$\mathrm{E}\left[(L\boldsymbol{\delta}) \cdot (L\boldsymbol{\delta})^T\right] = \mathrm{E}\left[L \cdot \boldsymbol{\delta} \cdot \boldsymbol{\delta}^T \cdot L^T\right] \tag{229}$$
$$= L \cdot \mathrm{E}\left[\boldsymbol{\delta} \cdot \boldsymbol{\delta}^T\right] \cdot L^T \tag{230}$$
$$= L \cdot \mathbf{\Sigma} \cdot L^T \tag{231}$$

So we can express the parameters of the transformed distribution:

$$y \cdot \tilde{x} \in \mathcal{N}\left(y \cdot x;\ \mathrm{Adj}_y \cdot \boldsymbol{\Sigma} \cdot \mathrm{Adj}_y^T\right) \tag{232}$$

Thus the mean of the transformed distribution is the transformed mean, and the covariance is mapped linearly by the adjoint.

8.3 Distribution of Inverse

Similarly, the distribution of the inverse transformation is easily computed:

$$\begin{aligned} \tilde{z} &\equiv \tilde{x}^{-1} && (233)\\ &= x^{-1} \cdot \exp\left(-\boldsymbol{\delta}\right) && (234)\\ &= \exp\left(-\mathrm{Adj}_{x^{-1}}\boldsymbol{\delta}\right) \cdot x^{-1} && (235)\\ \tilde{z} &\in \mathcal{N}\left(x^{-1};\ \mathrm{Adj}_{x^{-1}} \cdot \boldsymbol{\Sigma} \cdot \mathrm{Adj}_{x^{-1}}^T\right) && (236) \end{aligned}$$

8.4 Projection through Group Actions

Because the covariance is expressed in terms of tangent vectors (exponentiated and multiplied on the left-hand side), all the Jacobians given in the group descriptions above can be used to transform covariance (or information) matrices through arbitrary mappings.

8.5 Computing Moments from Samples

Given a Lie group G (with algebra $\mathfrak{g}$) and a set of N samples $x_i \in G$, we can estimate the mean and covariance iteratively. First, any of the samples provides an initial guess for the mean:

$$\mu_0 \leftarrow x_0 \tag{237}$$

Then, new estimates of the mean and covariance can be computed in terms of deviations in the tangent space:

$$\begin{aligned} v_{i,k} &\equiv \ln\left(x_i \cdot \mu_k^{-1}\right) && (238)\\ \boldsymbol{\Sigma}_k &\leftarrow \frac{1}{N}\sum_i \left[v_{i,k} \cdot v_{i,k}^T\right] && (239)\\ \mu_{k+1} &\leftarrow \exp\left(\frac{1}{N}\sum_i v_{i,k}\right) \cdot \mu_k && (240) \end{aligned}$$

Iterating these updates should rapidly converge (typically in two or three iterations). Replacing the factor $\frac{1}{N}$ with $\frac{1}{N-1}$ in Eq. 239 yields an unbiased estimate of the covariance.