Lie Groups for Computer Vision

Ethan Eade

1 Introduction

This document describes properties of transformation groups useful for computer vision, mainly intended as a reference for implementation. Lengthy derivations are omitted.

2 General Properties

2.1 Matrix Groups

A Lie group G is simultaneously a smooth differentiable manifold and a group. The Lie groups treated in this document are all real matrix groups: group elements are represented as matrices in $\mathbb{R}^{n\times n}$. The groups' multiplication and inversion operations are identically matrix multiplication and inversion. Because each group is represented by a specific subclass of non-singular $n \times n$ matrices, there are fewer than n^2 degrees of freedom.

2.2 Lie Algebra

Consider a Lie group G represented in $\mathbb{R}^{n\times n}$, with k degrees of freedom. The Lie algebra $\mathfrak{g}$ is the space of differential transformations – the *tangent space* – around the identity of G. This tangent space is a k-dimensional vector space with basis elements $\{G_1, \ldots, G_k\}$: the *generators*. Elements of $\mathfrak{g}$ are represented as matrices in $\mathbb{R}^{n\times n}$, but under addition and scalar multiplication, rather than matrix multiplication.

For such a Lie algebra $\mathfrak{g}$, we write the linear combination of generators $\{G_i\}$ specified by a vector of coefficients $\mathbf{c}$ as $\operatorname{alg}(\mathbf{c})$:

$$\operatorname{alg} : \mathbb{R}^k \rightarrow \mathfrak{g} \subset \mathbb{R}^{n\times n} \tag{1}$$

$$\operatorname{alg}(\mathbf{c}) \equiv \sum_{i=1}^{k} \mathbf{c}_i G_i \tag{2}$$

We denote the unique inverse of this linear combination by alg^{-1}. It might seem confusing that a *tangent vector* is in fact an $n \times n$ matrix, but it can always be thought of (and represented) as the vector of coefficients of the generators.

2.3 Exponential Map and Logarithm

The exponential map takes elements in the algebra to elements in the group. Intuitively speaking, it walks along the group manifold in the differential direction specified by the tangent vector in the algebra. For matrix groups the exponential map is simply matrix exponentiation:

$$\begin{aligned} \exp : \mathfrak{g} &\rightarrow \mathrm{G} && (3) \\ \exp(x) &= \mathbf{I} + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{i!}x^i + \cdots && (4) \end{aligned}$$

For several groups described below, the exponential map has a closed form. It is always a continuous map.

The inverse of the exponential map is the logarithm:

$$\exp(\log(X)) = X \tag{5}$$

The logarithm is usually not continuous everywhere, but is always continuous near the identity. Note that for most groups, including all groups with compact subgroups such as rotations, neither exp nor log is injective.

2.4 Interpolation on the Manifold

The exponential map and logarithm provides an intuitive method for interpolation or blending of transformations. Consider transformation $X, Y \in \mathrm{G}$ and an interpolation coefficient $t \in [0,1] \subset \mathbb{R}$. The function f blends the two transformations by moving steadily along the geodesic between them:

$$\begin{aligned} f : \mathrm{G} \times \mathrm{G} \times \mathbb{R} &\rightarrow \mathrm{G} && (6) \\ f(X, Y, t) &= \exp\left(t \cdot \log\left(Y \cdot X^{-1}\right)\right) \cdot X && (7) \\ \Longrightarrow f(X, Y, 0) &= X && (8) \\ \Longrightarrow f(X, Y, 1) &= Y && (9) \\ \Longrightarrow f\left(X, Y, \frac{1}{2}\right) \cdot X^{-1} &= Y \cdot f\left(X, Y, \frac{1}{2}\right)^{-1} && (10) \end{aligned}$$

2.5 Adjoint Representation

Consider tangent vectors $a, b \in \mathfrak{g}$ and a group element $X \in \mathrm{G}$. How can we choose b such that the following relation holds?

$$\exp(b) \cdot X = X \cdot \exp(a) \tag{11}$$

Right multiplying both sides by X^{-1} yields a conjugation by X:

$$\exp(b) = X \cdot \exp(a) \cdot X^{-1} \tag{12}$$

We could then compute b by taking the logarithm:

$$b = \log\left(X \cdot \exp(a) \cdot X^{-1}\right) \tag{13}$$

In fact, the identical result can be obtained by using the adjoint representation. A group $\mathrm{G} \subset \mathbb{R}^{n \times n}$ with k degrees of freedom has an isomorphic representation as the group of linear transformations on $\mathfrak{g}$, called the adjoint:

$$
\begin{aligned}
X &\in \mathrm{G} && (14)\\
a &\in \mathfrak{g} && (15)\\
\mathrm{Adj}_X : \mathfrak{g} &\to \mathfrak{g} && (16)\\
\mathrm{Adj}_X(a) &= X \cdot a \cdot X^{-1} \in \mathfrak{g} && (17)
\end{aligned}
$$

Elements of the adjoint representation are usually written as $k \times k$ matrices acting on the coefficient vectors of elements in $\mathfrak{g}$ by multiplication.

The adjoint representation preserves the group structure of G:

$$
\begin{aligned}
X, Y &\in \mathrm{G} && (18)\\
\mathrm{Adj}_{X \cdot Y} &= \mathrm{Adj}_X \cdot \mathrm{Adj}_Y && (19)\\
\mathrm{Adj}_{X^{-1}} &= \mathrm{Adj}_X^{-1} && (20)
\end{aligned}
$$

Returning to our motivating problem, we define b using the adjoint:

$$
\begin{aligned}
b &\equiv \mathrm{Adj}_X(a) && (21)\\
\implies \exp(b) &= X \cdot \exp(a) \cdot X^{-1} && (22)\\
\implies \exp(b) \cdot X &= X \cdot \exp(a) && (23)
\end{aligned}
$$

Thus the adjoint is effectively the Jacobian of the transformation of tangent vectors through elements of the group:

$$
\begin{aligned}
\mathbf{c} &\in \mathbb{R}^k && (24)\\
X &\in \mathrm{G} && (25)\\
f : \mathrm{G} \times \mathbb{R}^k &\to \mathbb{R}^k && (26)\\
f(X, \mathbf{c}) &= \mathrm{alg}^{-1}\left(\log\left(X \cdot \exp(\mathrm{alg}(c)) \cdot X^{-1}\right)\right) && (27)\\
\frac{\partial f}{\partial \mathbf{c}}\Big|_{\mathbf{c}=0} &= \mathrm{Adj}_X && (28)
\end{aligned}
$$

2.6 Group Action on $\mathbb{R}^n$

Given the matrix representation of G in $\mathbb{R}^{n \times n}$, there is a natural action on the vector space $\mathbb{R}^n$ (equivalently the projective space $\mathbb{P}^{n-1}$) by multiplication:

$$
\begin{aligned}
X &\in \mathrm{G} && (29)\\
X : \mathbb{R}^n &\to \mathbb{R}^n && (30)\\
X(\mathbf{v}) &= X \cdot \mathbf{v} && (31)
\end{aligned}
$$

For the groups described below, this group action by matrix multiplication yields a transformation on points or lines in 2D or 3D Euclidean or projective space. For example, the group action of an element of SE(2) on $\mathbb{P}^2$ (the 2D plane as homogeneous coordinates in $\mathbb{R}^3$) is a rotation and translation of the plane coordinates.

The Jacobian of this action by the group differentials around the identity is trivially computed using the k generators of the algebra:

$$
\begin{aligned}
\mathbf{p} &\in \mathbb{R}^n && (32)\\
\mathbf{c} &\in \mathbb{R}^k && (33)\\
f(\mathbf{c}, p) &\equiv \exp(\mathrm{alg}(\mathbf{c})) \cdot \mathbf{p} && (34)\\
\frac{\partial f}{\partial \mathbf{c}}\Big|_{\mathbf{c}=0} &= \left(\begin{array}{c|c|c|c} G_1 \cdot \mathbf{p} & G_2 \cdot \mathbf{p} & \cdots & G_k \cdot \mathbf{p} \end{array} \right) \in \mathbb{R}^{n \times k} && (35)
\end{aligned}
$$

3 SO(2)

3.1 Description

SO(2) is the group of rotations in the 2D plane. It has one degree of freedom: angle of rotation. The group is commutative. The inverse is given by the transpose:

$$
\begin{aligned}
X &\in \mathrm{SO}(2) \subset \mathbb{R}^{2\times 2} && (36)\\
X^{-1} &= X^T && (37)
\end{aligned}
$$

3.2 Lie Algebra

The Lie algebra $\mathfrak{so}(2)$ is generated by one antisymmetric element, corresponding to differential rotation:

$$
G_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad (38)
$$

3.3 Exponential Map

The exponential map from $\mathfrak{so}(2)$ to SO(2) is simply a 2D rotation:

$$
\exp(\mathrm{alg}(\theta)) = \exp \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad (39)
$$

The logarithm is trivially computed from an element of SO(2).

3.4 Adjoint Representation

The adjoint representation of SO(2) is trivial:

$$
\begin{aligned}
X &= \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in \mathrm{SO}(2), a^2 + b^2 = 1 && (40)\\
\mathrm{Adj}_X(\mathrm{alg}(\theta)) &= X \cdot \mathrm{alg}(\theta) \cdot X^{-1} && (41)\\
&= \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} && (42)\\
&= \mathrm{alg}(\theta) && (43)\\
\implies \mathrm{Adj}_X &= \mathbf{I} && (44)
\end{aligned}
$$

4 SE(2)

4.1 Description

SE(2) is the group of rigid transformations in the 2D plane, the semi-direct product $SO(2) \ltimes \mathbb{R}^2$. It has three degrees of freedom: two for translation and one for rotation. Subgroups include SO(2).

$$
\begin{aligned}
\mathbf{R} &\in \mathrm{SO}(2) && (45)\\
\mathbf{t} &\in \mathbb{R}^2 && (46)\\
X = \left(\begin{array}{c|c} \mathbf{R} & \mathbf{t} \\ \hline \mathbf{0} & 1 \end{array}\right) &\in \mathrm{SE}(2) \subset \mathbb{R}^{3\times 3} && (47)\\
X^{-1} &= \left(\begin{array}{c|c} \mathbf{R}^T & -\mathbf{R}^T\mathbf{t} \\ \hline \mathbf{0} & 1 \end{array}\right) && (48)
\end{aligned}
$$

4.2 Lie Algebra

The Lie algebra $\mathfrak{se}(2)$ has three generators:

$$
G_1 = \left(\begin{array}{cc|c} 0 & 0 & 1 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right), G_2 = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 0 & 0 \end{array}\right), G_3 = \left(\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \tag{49}
$$

4.3 Exponential Map

The exponential map from $\mathfrak{se}(2)$ to SE(2) has a closed form:

$$
\begin{aligned}
v = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} &\in \mathbb{R}^3 && (50)\\
\mathbf{R} &\equiv \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} && (51)\\
\mathbf{V} &= \begin{pmatrix} \frac{\sin\theta}{\theta} & -\frac{1-\cos\theta}{\theta} \\ \frac{1-\cos\theta}{\theta} & \frac{\sin\theta}{\theta} \end{pmatrix} && (52)\\
\exp\left(\mathrm{alg}\left(v\right)\right) = \exp\left(\begin{array}{cc|c} 0 & -\theta & x \\ \theta & 0 & y \\ \hline 0 & 0 & 0 \end{array}\right) &= \left(\begin{array}{c|c} \mathbf{R} & \mathbf{V}\cdot\begin{pmatrix} x \\ y \end{pmatrix} \\ \hline \mathbf{0} & 1 \end{array}\right) && (53)
\end{aligned}
$$

The elements of $\mathbf{V}$ should be calculated with Taylor series when θ is small (see Section 11).

4.4 Adjoint Representation

$$
\begin{aligned}
X &= \left(\begin{array}{c|c} \mathbf{R} & \mathbf{t} \\ \hline \mathbf{0} & 1 \end{array}\right) \in \mathrm{SE}(2) && (54)\\
\mathrm{Adj}_X &= \left(\begin{array}{c|c} \mathbf{R} & \begin{pmatrix} \mathbf{t}_1 \\ -\mathbf{t}_0 \end{pmatrix} \\ \hline 0 & 1 \end{array}\right) && (55)
\end{aligned}
$$

5 Sim(2)

5.1 Description

Sim(2) is the group of orientation-preserving similarity transformations in the 2D plane, the semi-direct product $\mathrm{SE}(2) \rtimes \mathbb{R}^*$. It has four degrees of freedom: two for translation, one for rotation, and one for scale. Subgroups include SE(2) and $\mathbb{R}^*$.

$$X = \left(\begin{array}{c|c} \mathbf{R} & \mathbf{t} \\ \hline \mathbf{0} & s^{-1} \end{array}\right) \quad \in \quad \mathrm{Sim}(2) \subset \mathbb{R}^{3\times 3} \tag{56}$$

$$X^{-1} \quad = \quad \left(\begin{array}{c|c} \mathbf{R}^T & -s\mathbf{R}^T\mathbf{t} \\ \hline \mathbf{0} & s \end{array}\right) \tag{57}$$

5.2 Lie Algebra

The Lie algebra $\mathfrak{sim}(2)$ has four generators:

$$G_1 = \left(\begin{array}{cc|c} 0 & 0 & 1 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right), G_2 = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 0 & 0 \end{array}\right), G_3 = \left(\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right), G_4 = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & -1 \end{array}\right) \tag{58}$$

5.3 Exponential Map

The exponential map from $\mathfrak{sim}(2)$ to $\mathrm{Sim}(2)$ has a closed form:

$$v = \begin{pmatrix} x \\ y \\ \theta \\ \lambda \end{pmatrix} \quad \in \quad \mathbb{R}^4 \tag{59}$$

$$\mathbf{R} \quad \equiv \quad \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{60}$$

$$A \quad \equiv \quad \frac{\sin\theta}{\theta} \tag{61}$$

$$B \quad \equiv \quad \frac{1-\cos\theta}{\theta^2} \tag{62}$$

$$C \quad \equiv \quad \frac{\theta - \sin\theta}{\theta^3} \tag{63}$$

$$\alpha \quad \equiv \quad \frac{\lambda^2}{\lambda^2 + \theta^2} \tag{64}$$

$$s \quad \equiv \quad e^{\lambda}$$

$$X \quad \equiv \quad \alpha\left(\frac{1-s^{-1}}{\lambda}\right) + (1-\alpha)\,(A - \lambda B) \tag{65}$$

$$Y \quad \equiv \quad \alpha\left(\frac{s^{-1}-1+\lambda}{\lambda^2}\right) + (1-\alpha)\,(B - \lambda C) \tag{66}$$

$$\mathbf{V} \quad = \quad \begin{pmatrix} X & -\theta Y \\ \theta Y & X \end{pmatrix} \tag{67}$$

$$\exp\left(\mathrm{alg}\left(v\right)\right) = \exp\left(\begin{array}{cc|c} 0 & -\theta & x \\ \theta & 0 & y \\ \hline 0 & 0 & -\lambda \end{array}\right) \quad = \quad \left(\begin{array}{c|c} \mathbf{R} & \mathbf{V}\cdot\begin{pmatrix} x \\ y \end{pmatrix} \\ \hline \mathbf{0} & s^{-1} \end{array}\right) \tag{68}$$

The elements of $\mathbf{V}$ should be calculated with Taylor series when θ or λ is small (see Section 11).

5.4 Adjoint Representation

$$X = \left(\begin{array}{c|c} \mathbf{R} & \mathbf{t} \\ \hline \mathbf{0} & s^{-1} \end{array}\right) \in \mathrm{Sim}(2) \tag{69}$$

$$\mathrm{Adj}_X = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{s\mathbf{R}} & s \cdot \begin{pmatrix} \mathbf{t}_1 & -\mathbf{t}_0 \\ -\mathbf{t}_0 & -\mathbf{t}_1 \end{pmatrix} \\ \hline 0 & 0 & 1 \quad 0 \\ 0 & 0 & 0 \quad 1 \end{array}\right) \tag{70}$$

6 Aff(2)

6.1 Description

Aff(2) is the group of affine transformations on the 2D plane. It has six degrees of freedom: two for translation, one for rotation, one for scale, one for stretch and one for shear. Subgroups include Sim(2).

$$X = \left(\begin{array}{c|c} \mathbf{A} & \mathbf{t} \\ \hline \mathbf{0} & 1 \end{array}\right) \in \mathrm{Aff}(2) \subset \mathbb{R}^{3\times 3} \tag{71}$$

$$X^{-1} = \left(\begin{array}{c|c} \mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{t} \\ \hline \mathbf{0} & 1 \end{array}\right) \tag{72}$$

6.2 Lie Algebra

The Lie algebra $\mathfrak{aff}(2)$ has six generators:

$$G_1 = \left(\begin{array}{cc|c} 0 & 0 & 1 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right), \quad G_2 = \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 0 & 0 \end{array}\right), \quad G_3 = \left(\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right), \tag{73}$$

$$G_4 = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right), \quad G_5 = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \quad G_6 = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \tag{74}$$

6.3 Exponential Map

The exponential map from $\mathfrak{aff}(2)$ to Aff(2) has no closed form. It can be computed by any general matrix exponential routine. The same is true for the logarithm.

6.4 Adjoint Representation

Let

$$X = \left(\begin{array}{c|c} \mathbf{A} & \mathbf{t} \\ \hline \mathbf{0} & 1 \end{array}\right) = \left(\begin{array}{cc|c} a & b & x \\ c & d & y \\ \hline 0 & 0 & 1 \end{array}\right) \quad \in \quad \mathrm{Aff}(2) \tag{75}$$

$$\mathbf{E} \equiv \left(\begin{array}{cc|cccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{array}\right) \in \mathbb{R}^{6\times 6} \tag{76}$$

$$f \equiv \frac{1}{ad - bc} \tag{77}$$

$$\mathbf{C} \equiv \left(\begin{array}{c|c} a\cdot\mathbf{A}^{-T} & b\cdot\mathbf{A}^{-T} \\ \hline c\cdot\mathbf{A}^{-T} & d\cdot\mathbf{A}^{-T} \end{array}\right) \in \mathbb{R}^{4\times 4} \tag{78}$$

$$= f\begin{pmatrix} ad & -ac & bd & -bc \\ -ab & a^2 & -b^2 & ab \\ cd & -c^2 & d^2 & -cd \\ -bc & ac & -bd & ad \end{pmatrix} \tag{79}$$

$$\mathbf{T} \equiv \begin{pmatrix} x & y & 0 & 0 \\ 0 & 0 & x & y \end{pmatrix} \in \mathbb{R}^{2\times 4} \tag{80}$$

Then

$$\mathrm{Adj}_X = \mathbf{E}^T \left(\begin{array}{c|c} \mathbf{A} & -\mathbf{T}\cdot\mathbf{C} \\ \hline \mathbf{0} & \mathbf{C} \end{array}\right) \mathbf{E} \tag{81}$$

Writing out the product explicitly:

$$\mathrm{Adj}_X = \left(\begin{array}{cc|cccc} a & b & fy\left(a^2+b^2\right) - fx\left(ac+bd\right) & -x & fy\left(2ab\right) - fx\left(ad+bc\right) & fx\left(ac-bd\right) - fy\left(a^2-b^2\right) \\ c & d & fy\left(ac+bd\right) - fx\left(c^2+d^2\right) & -y & fy\left(ad+bc\right) - fx\left(2cd\right) & fx\left(c^2-d^2\right) - fy\left(ac-bd\right) \\ \hline 0 & 0 & \frac{f}{2}\left(a^2+b^2+c^2+d^2\right) & 0 & f\left(ab+cd\right) & \frac{f}{2}\left(-a^2+b^2-c^2+d^2\right) \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & f\left(ac+bd\right) & 0 & f\left(ad+bc\right) & f\left(bd-ac\right) \\ 0 & 0 & \frac{f}{2}\left(-a^2-b^2+c^2+d^2\right) & 0 & f\left(cd-ab\right) & \frac{f}{2}\left(a^2-b^2-c^2+d^2\right) \end{array}\right) \tag{82}$$

7 SL(3)

7.1 Description

SL(3) is the group of unit-determinant linear transformations, representing among other things homographies on the 2D projective plane. It has eight degrees of freedom: two for translation, one for rotation, one for scale, one for stretch, one for shear, and two for perspective change. Subgroups include Aff(2) and SO(3).

$$\mathbf{H} \in \mathrm{SL}(3) \subset \mathbb{R}^{3\times 3} \tag{83}$$

$$\det\left(\mathbf{H}\right) = 1 \tag{84}$$

7.2 Lie Algebra

The Lie algebra $\mathfrak{sl}(3)$ has eight generators, all with zero trace:

$$G_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad G_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{85}$$

$$G_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad G_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad G_6 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{86}$$

$$G_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad G_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{87}$$

7.3 Exponential Map

The exponential map from $\mathfrak{sl}(3)$ to SL(3) has no closed form. It can be computed by any general matrix exponential routine. The same is true for the logarithm. Note that the exponential of any traceless square matrix is a matrix with unit determinant.

7.4 Adjoint Representation

First we treat elements of $\mathfrak{sl}(3)$, which are 3×3 matrices, as 9-vectors, writing the entries in row-major order. Then, for $h \in \mathfrak{sl}(3)$ and $\mathbf{H} \in \mathrm{SL}(3)$, the conjugation $\mathbf{H} \cdot h \cdot \mathbf{H}^{-1}$ can be expressed as a linear mapping $\mathbf{C_H}$ on the elements of h. Pre- and post- applying matrix representations of alg and alg^{-1} respectively then gives the adjoint representation.

Let

$$[\mathrm{alg}] \equiv \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{9\times 8} \tag{88}$$

$$\left[\mathrm{alg}^{-1}\right] \equiv \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \in \mathbb{R}^{8\times 9} \tag{89}$$

$$\mathbf{C_H} \equiv \left(\begin{array}{c|c|c} \mathbf{H}_{11}\mathbf{H}^{-T} & \mathbf{H}_{12}\mathbf{H}^{-T} & \mathbf{H}_{13}\mathbf{H}^{-T} \\ \hline \mathbf{H}_{21}\mathbf{H}^{-T} & \mathbf{H}_{22}\mathbf{H}^{-T} & \mathbf{H}_{23}\mathbf{H}^{-T} \\ \hline \mathbf{H}_{31}\mathbf{H}^{-T} & \mathbf{H}_{32}\mathbf{H}^{-T} & \mathbf{H}_{33}\mathbf{H}^{-T} \end{array}\right) \tag{90}$$

Then

$$\mathrm{Adj}_{\mathbf{H}} = \left[\mathrm{alg}^{-1}\right] \cdot \mathbf{C}_{\mathbf{H}} \cdot [\mathrm{alg}] \tag{91}$$

8 SO(3)

8.1 Description

SO(3) is the group of rotations in 3D space, represented by $3x3$ orthogonal matrices with unit determinant. It has three degrees of freedom: one for each differential rotation axis. The inverse is given by the transpose:

$$\begin{aligned}
\mathbf{R} &\in \mathrm{SO}(3) \subset \mathbb{R}^{3\times 3} && (92)\\
\mathbf{R}^{-1} &= \mathbf{R}^{T} && (93)\\
\det(\mathbf{R}) &= 1 && (94)
\end{aligned}$$

8.2 Lie Algebra

The Lie algebra $\mathfrak{so}(3)$ is the set of antisymmetric 3×3 matrices, generated by the differential rotations about each axis:

$$G_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, G_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, G_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{95}$$

The mapping $\mathrm{alg} : \mathbb{R}^3 \to \mathfrak{so}(3)$ sends 3-vectors to their skew matrix:

$$\begin{aligned}
\boldsymbol{\omega} &\equiv \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 && (96)\\
\mathrm{alg}(\boldsymbol{\omega}) &= \boldsymbol{\omega}_{\times} && (97)\\
&= \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} && (98)
\end{aligned}$$

8.3 Exponential Map

The exponential map from $\mathfrak{so}(3)$ to SO(3) has a closed form (also called the Rodrigues formula). The tangent vector $\boldsymbol{\omega}$ can be interpreted as an axis-angle representation of rotation: its exponential is the rotation around the axis $\boldsymbol{\omega}/\left\|\boldsymbol{\omega}\right\|$ by $\left\|\boldsymbol{\omega}\right\|$ radians:

$$\begin{aligned}
\boldsymbol{\omega} &\in \mathbb{R}^3 && (99)\\
\theta &\equiv \sqrt{\boldsymbol{\omega}^T \boldsymbol{\omega}} && (100)\\
\exp(\mathrm{alg}\boldsymbol{\omega}) &= \exp(\boldsymbol{\omega}_{\times}) && (101)\\
&= \mathbf{I} + \boldsymbol{\omega}_{\times} + \frac{1}{2!}\boldsymbol{\omega}_{\times}^2 + \frac{1}{3!}\boldsymbol{\omega}_{\times}^3 + \ldots && (102)\\
&= \mathbf{I} + \left(\frac{\sin\theta}{\theta}\right)\boldsymbol{\omega}_{\times} + \left(\frac{1-\cos\theta}{\theta^2}\right)\boldsymbol{\omega}_{\times}^2 && (103)
\end{aligned}$$

The higher-order terms in Eq. 102 collapse because $\boldsymbol{\omega}_\times^3 = -\theta^2 \boldsymbol{\omega}_\times$. The coefficients of $\mathbf{R}$ should be calculated with Taylor series when θ is small (see Section 11).

Given a rotation matrix $\mathbf{R} \in \mathrm{SO}(3)$, the logarithm can be computed by first determining $\cos\theta = \frac{1}{2}(\operatorname{tr}(\mathbf{R}) - 1)$, and then computing $\boldsymbol{\omega}$ from symmetric differences (see the second term of Eq. 103).

8.4 Adjoint Representation

The adjoint representation of SO(3) is actually identical to the rotation matrix representation due to properties of the cross product:

$$
\begin{aligned}
\mathbf{R} &\in \mathrm{SO}(3) && (104)\\
\mathbf{a}, \mathbf{b} &\in \mathbb{R}^3 && (105)\\
\left(\mathbf{R} \cdot \mathbf{a}_\times \cdot \mathbf{R}^T\right) \cdot \mathbf{b} &= \mathbf{R} \cdot \left(\mathbf{a} \times \mathbf{R}^T \cdot \mathbf{b}\right) && (106)\\
&= (\mathbf{R} \cdot \mathbf{a}) \times \mathbf{b} && (107)\\
&= (\mathbf{R} \cdot \mathbf{a})_\times \cdot \mathbf{b} && (108)\\
\implies \mathbf{R} \cdot \mathbf{a}_\times \cdot \mathbf{R}^T &= (\mathbf{R} \cdot \mathbf{a})_\times && (109)\\
\implies \mathrm{Adj}_{\mathbf{R}} &= \mathbf{R} && (110)
\end{aligned}
$$

9 SE(3)

9.1 Description

SE(3) is the group of rigid transformations in 3D space, the semi-direct product $\mathrm{SO}(3) \ltimes \mathbb{R}^3$. It has six degrees of freedom: three for translation and three for rotation. Subgroups include SE(2) and SO(3).

$$
\begin{aligned}
\mathbf{R} &\in \mathrm{SO}(3) && (111)\\
\mathbf{t} &\in \mathbb{R}^3 && (112)\\
X &= \left(\begin{array}{c|c} \mathbf{R} & \mathbf{t} \\ \hline \mathbf{0} & 1 \end{array}\right) \in \mathrm{SE}(3) \subset \mathbb{R}^{4\times 4} && (113)\\
X^{-1} &= \left(\begin{array}{c|c} \mathbf{R}^T & -\mathbf{R}^T\mathbf{t} \\ \hline \mathbf{0} & 1 \end{array}\right) && (114)
\end{aligned}
$$

9.2 Lie Algebra

The Lie algebra $\mathfrak{se}(3)$ has six generators:

$$
G_1 = \left(\begin{array}{ccc|c} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right), \quad G_2 = \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right), \quad G_3 = \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \end{array}\right) \qquad (115)
$$

$$
G_4 = \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right), \quad G_5 = \left(\begin{array}{ccc|c} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right), \quad G_6 = \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right) \qquad (116)
$$

Thus the mapping $\mathrm{alg} : \mathbb{R}^3 \to \mathfrak{se}(3)$:

$$\begin{aligned} \mathbf{u}, \boldsymbol{\omega} &\in \mathbb{R}^3 && (117) \\ \mathrm{alg}\begin{pmatrix} \mathbf{u} \\ \boldsymbol{\omega} \end{pmatrix} &= \left(\begin{array}{c|c} \boldsymbol{\omega}_\times & \mathbf{u} \\ \hline 0 & 0 \end{array}\right) && (118) \end{aligned}$$

9.3 Exponential Map

The exponential map from $\mathfrak{se}(3)$ to SE(3) has a closed form:

$$\begin{aligned} \mathbf{u}, \boldsymbol{\omega} &\in \mathbb{R}^3 && (119) \\ \theta &\equiv \sqrt{\boldsymbol{\omega}^T \boldsymbol{\omega}} && (120) \\ \exp\left(\mathrm{alg}\begin{pmatrix} \mathbf{u} \\ \boldsymbol{\omega} \end{pmatrix}\right) &= \exp\left(\begin{array}{c|c} \boldsymbol{\omega}_\times & \mathbf{u} \\ \hline 0 & 0 \end{array}\right) && (121) \\ &= \mathbf{I} + \left(\begin{array}{c|c} \boldsymbol{\omega}_\times & \mathbf{u} \\ \hline 0 & 0 \end{array}\right) + \frac{1}{2!}\left(\begin{array}{c|c} \boldsymbol{\omega}_\times^2 & \boldsymbol{\omega}_\times \mathbf{u} \\ \hline 0 & 0 \end{array}\right) + \frac{1}{3!}\left(\begin{array}{c|c} \boldsymbol{\omega}_\times^2 & \boldsymbol{\omega}_\times^2 \mathbf{u} \\ \hline 0 & 0 \end{array}\right) + \ldots && (122) \\ &= \left(\begin{array}{c|c} \exp\left(\boldsymbol{\omega}_\times\right) & \mathbf{V} \cdot \mathbf{u} \\ \hline 0 & 1 \end{array}\right) && (123) \\ \mathbf{V} &\equiv \mathbf{I} + \left(\frac{1 - \cos\theta}{\theta^2}\right)\boldsymbol{\omega}_\times + \left(\frac{\theta - \sin\theta}{\theta^3}\right)\boldsymbol{\omega}_\times^2 && (124) \end{aligned}$$

Note that the rotation block is computed according to Eq. 103. The coefficients of $\mathbf{V}$ should be calculated with Taylor series when θ is small (see Section 11).

The inverse of $\mathbf{V}$ can also be written in closed form:

$$\mathbf{V}^{-1} = \mathbf{I} - \frac{1}{2}\boldsymbol{\omega}_\times + \frac{1}{\theta^2}\left(1 - \frac{\theta \sin\theta}{2\left(1 - \cos\theta\right)}\right)\boldsymbol{\omega}_\times^2 \tag{125}$$

The logarithm of $\left(\begin{array}{c|c} \mathbf{R} & \mathbf{t} \\ \hline \mathbf{0} & 1 \end{array}\right) \in \mathrm{SE}(3)$ can be determined by first computing $\boldsymbol{\omega} = \mathrm{alg}^{-1}\left(\log\left(\mathbf{R}\right)\right)$, then computing $\mathbf{u} = \mathbf{V}^{-1} \cdot \mathbf{t}$.

9.4 Adjoint Representation

$$\begin{aligned} X &= \left(\begin{array}{c|c} \mathbf{R} & \mathbf{t} \\ \hline \mathbf{0} & 1 \end{array}\right) \in \mathrm{SE}(3) && (126) \\ \mathrm{Adj}_X &= \left(\begin{array}{c|c} \mathbf{R} & \mathbf{t}_\times \mathbf{R} \\ \hline \mathbf{0} & \mathbf{R} \end{array}\right) \in \mathbb{R}^{6\times 6} && (127) \end{aligned}$$

10 Sim(3)

10.1 Description

Sim(3) is the group of similarity transformations in 3D space, the semi-direct product $\mathrm{SE}(3) \rtimes \mathbb{R}^*$. It has seven degrees of freedom: three for translation, three for rotation, and one for scale. Subgroups include Sim(2) and SE(3).

$$
\begin{aligned}
\mathbf{R} &\in \mathrm{SO}(3) && (128)\\
\mathbf{t} &\in \mathbb{R}^3 && (129)\\
s &\in \mathbb{R}^+ && (130)\\
X &= \left(\begin{array}{c|c} \mathbf{R} & \mathbf{t} \\ \hline \mathbf{0} & s^{-1} \end{array}\right) \in \mathrm{Sim}(3) \subset \mathbb{R}^{4\times 4} && (131)\\
X^{-1} &= \left(\begin{array}{c|c} \mathbf{R}^T & -s\mathbf{R}^T\mathbf{t} \\ \hline \mathbf{0} & s \end{array}\right) && (132)
\end{aligned}
$$

10.2 Lie Algebra

The Lie algebra $\mathfrak{sim}(3)$ has seven generators:

$$
G_1 = \left(\begin{array}{ccc|c} 0&0&0&1\\0&0&0&0\\0&0&0&0\\ \hline 0&0&0&0 \end{array}\right), \quad
G_2 = \left(\begin{array}{ccc|c} 0&0&0&0\\0&0&0&1\\0&0&0&0\\ \hline 0&0&0&0 \end{array}\right), \quad
G_3 = \left(\begin{array}{ccc|c} 0&0&0&0\\0&0&0&0\\0&0&0&1\\ \hline 0&0&0&0 \end{array}\right) \tag{133}
$$

$$
G_4 = \left(\begin{array}{ccc|c} 0&0&0&0\\0&0&1&0\\0&-1&0&0\\ \hline 0&0&0&0 \end{array}\right), \quad
G_5 = \left(\begin{array}{ccc|c} 0&0&-1&0\\0&0&0&0\\1&0&0&0\\ \hline 0&0&0&0 \end{array}\right), \quad
G_6 = \left(\begin{array}{ccc|c} 0&1&0&0\\-1&0&0&0\\0&0&0&0\\ \hline 0&0&0&0 \end{array}\right), \tag{134}
$$

$$
G_7 = \left(\begin{array}{ccc|c} 0&0&0&0\\0&0&0&0\\0&0&0&0\\ \hline 0&0&0&-1 \end{array}\right) \tag{135}
$$

10.3 Exponential Map

The exponential map from $\mathfrak{sim}(3)$ to $\mathrm{Sim}(3)$ has a closed form:

$$
\begin{aligned}
v = \begin{pmatrix} \mathbf{u} \\ \boldsymbol{\omega} \\ \lambda \end{pmatrix} \quad &\in \quad \mathbb{R}^7 && (136)\\
\theta \quad &\equiv \quad \sqrt{\boldsymbol{\omega}^T \boldsymbol{\omega}} && (137)\\
A \quad &\equiv \quad \frac{\sin\theta}{\theta} && (138)\\
B \quad &\equiv \quad \frac{1-\cos\theta}{\theta^2} && (139)\\
C \quad &\equiv \quad \frac{1-A}{\theta^2} && (140)\\
D \quad &\equiv \quad \frac{\frac{1}{2}-B}{\theta^2} && (141)\\
s^{-1} \quad &\equiv \quad e^{-\lambda} && (142)\\
\alpha \quad &\equiv \quad \frac{\lambda^2}{\lambda^2+\theta^2} && (143)\\
\beta \quad &\equiv \quad \frac{s^{-1}-1+\lambda}{\lambda^2} && (144)\\
\gamma \quad &\equiv \quad \frac{\frac{1}{2}-\beta}{\lambda} && (145)\\
X \quad &\equiv \quad \frac{1-s^{-1}}{\lambda} && (146)\\
Y \quad &\equiv \quad \alpha\cdot\beta + (1-\alpha)\cdot(B-\lambda C) && (147)\\
Z \quad &\equiv \quad \alpha\cdot\gamma + (1-\alpha)\cdot(C-\lambda D) && (148)\\
\mathbf{R} \quad &\equiv \quad \mathbf{I} + A\boldsymbol{\omega}_\times + B\boldsymbol{\omega}_\times^2 && (149)\\
\mathbf{V} \quad &\equiv \quad X\mathbf{I} + Y\boldsymbol{\omega}_\times + Z\boldsymbol{\omega}_\times^2 && (150)\\
\exp\left(\mathrm{alg}\left(v\right)\right) = \exp\left(\begin{array}{c|c} \boldsymbol{\omega}_\times & \mathbf{u} \\ \hline 0 & -\lambda \end{array}\right) \quad &= \quad \left(\begin{array}{c|c} \mathbf{R} & \mathbf{V}\cdot\mathbf{u} \\ \hline \mathbf{0} & s^{-1} \end{array}\right) && (151)
\end{aligned}
$$

The coefficients of $\mathbf{R}$ and $\mathbf{V}$ should be calculated with Taylor series when θ or λ is small (see Section 11).

10.4 Adjoint Representation

$$
\begin{aligned}
X \quad &= \quad \left(\begin{array}{c|c} \mathbf{R} & \mathbf{t} \\ \hline \mathbf{0} & s^{-1} \end{array}\right) \in \mathrm{Sim}(3) && (152)\\
\mathrm{Adj}_X \quad &= \quad \left(\begin{array}{c|c|c} s\mathbf{R} & s\mathbf{t}_\times\mathbf{R} & -s\mathbf{t} \\ \hline \mathbf{0} & \mathbf{R} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & 1 \end{array}\right) \in \mathbb{R}^{7\times 7} && (153)
\end{aligned}
$$

11 Taylor Series

These are Taylor series for coefficients in the above equations, for when the parameters are near zero:

$$\frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{6} + \frac{\theta^4}{120} - \frac{\theta^6}{5040} + O\left(\theta^8\right) \tag{154}$$

$$\approx 1 - \frac{\theta^2}{6}\left(1 - \frac{\theta^2}{20}\left(1 - \frac{\theta^2}{42}\right)\right) \tag{155}$$

$$\frac{1-\cos\theta}{\theta^2} = \frac{1}{2} - \frac{\theta^2}{24} + \frac{\theta^4}{720} - \frac{\theta^6}{40320} + O\left(\theta^8\right) \tag{156}$$

$$\approx \frac{1}{2}\left(1 - \frac{\theta^2}{12}\left(1 - \frac{\theta^2}{30}\left(1 - \frac{\theta^2}{56}\right)\right)\right) \tag{157}$$

$$\frac{\theta - \sin\theta}{\theta^3} = \frac{1}{6} - \frac{\theta^2}{120} + \frac{\theta^4}{5040} - \frac{\theta^6}{362880} + O\left(\theta^8\right) \tag{158}$$

$$\approx \frac{1}{6}\left(1 - \frac{\theta^2}{20}\left(1 - \frac{\theta^2}{42}\left(1 - \frac{\theta^2}{72}\right)\right)\right) \tag{159}$$

$$\frac{\frac{1}{2} - \frac{1-\cos\theta}{\theta^2}}{\theta^2} = \frac{1}{24} - \frac{\theta^2}{720} + \frac{\theta^4}{40320} - \frac{\theta^6}{3628800} + O\left(\theta^8\right) \tag{160}$$

$$\approx \frac{1}{24}\left(1 - \frac{\theta^2}{30}\left(1 - \frac{\theta^2}{56}\left(1 - \frac{\theta^2}{90}\right)\right)\right) \tag{161}$$

$$\frac{1}{\theta^2}\left(1 - \frac{\theta\sin\theta}{2\left(1-\cos\theta\right)}\right) = \frac{1}{12} + \frac{\theta^2}{720} + \frac{\theta^4}{30240} + \frac{\theta^6}{1209600} + O\left(\theta^8\right) \tag{162}$$

$$\approx \frac{1}{12}\left(1 + \frac{\theta^2}{60}\left(1 + \frac{\theta^2}{42}\left(1 + \frac{\theta^2}{40}\right)\right)\right) \tag{163}$$

$$\frac{1-e^{-\lambda}}{\lambda} = 1 - \frac{\lambda}{2} + \frac{\lambda^2}{6} - \frac{\lambda^3}{24} + O\left(\lambda^4\right) \tag{164}$$

$$\approx 1 - \frac{\lambda}{2}\left(1 - \frac{\lambda}{3}\left(1 - \frac{\lambda}{4}\right)\right) \tag{165}$$

$$\frac{e^{-\lambda} - 1 + \lambda}{\lambda^2} = \frac{1}{2} - \frac{\lambda}{6} + \frac{\lambda^2}{24} - \frac{\lambda^3}{120} + O\left(\lambda^4\right) \tag{166}$$

$$\approx \frac{1}{2}\left(1 - \frac{\lambda}{3}\left(1 - \frac{\lambda}{4}\left(1 - \frac{\lambda}{5}\right)\right)\right) \tag{167}$$

$$\frac{\frac{1}{2} - \frac{e^{-\lambda}-1+\lambda}{\lambda^2}}{\lambda} = \frac{1}{6} - \frac{\lambda}{24} + \frac{\lambda^2}{120} - \frac{\lambda^3}{720} + O\left(\lambda^4\right) \tag{168}$$

$$\approx \frac{1}{6}\left(1 - \frac{\lambda}{4}\left(1 - \frac{\lambda}{5}\left(1 - \frac{\lambda}{6}\right)\right)\right) \tag{169}$$